Households, Auctioneers, and Aggregation*

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Abstract

We examine aggregation in the neoclassical growth model with aggregate shocks and uninsurable employment risk, as well as related environments. We introduce a Walrasian auctioneer whose job is to report to households all possible state-contingent future prices. Households take these as given when forming expectations and making optimal consumption / savings decisions, and the auctioneer adjusts her forecasts until markets clear. This natural dichotomy between the households and the auctioneer allows us to study each problem in isolation as well as to discuss the intersection. On the household side, we separate an explicit expression for the linear permanent income component of savings from a well-behaved nonlinear adjustment arising from precautionary behavior and incomplete markets. Equipped with this decomposition, we then study how economies aggregate in the presence of various auctioneer types that are popular in the literature. The steady-state auctioneer of Huggett (1997) and Aiyagari (1994) offers a paper-and-pencil analysis of aggregation that provides a bound on more complex environments. We provide an economic interpretation of the regression coefficients and explain the lack of time variation in the auctioneer of Krusell and Smith (1998). We also introduce a new numerical method which uses the empirical distribution of auctioneer forecasts to substantially improve solution accuracy in cases where the standard coefficient of determination and other well-known statistics prove to be misleading.

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1 Introduction

Dynamic models with non-insurable idiosyncratic shocks have become standard in macroeconomics. Nearly all of the critical results—transmission of idiosyncratic and aggregate shocks, evolution of the distribution of wealth, welfare, etc.—rely heavily on the extent to which the economy aggregates [Krusell and Smith (2006), Heathcote, Storesletten, and Violante (2009), Guvenen (2011), Krueger, Mitman, and Perri (2016)]. Despite the importance of aggregation, few papers have sought to develop the foundations of such properties. Models with rich heterogeneity primarily rely on quantitative analysis—making the assessment of aggregation difficult [e.g., Kaplan, Moll, and Violante (2018)]—while tractable models often invoke assumptions which serve to simplify aggregation [e.g., Bilbiie (2008)]. We study aggregation, and the insights provided therein, in well-known incomplete markets models that permit substantial heterogeneity.

The challenge associated with studying aggregation in these models is well known. When aggregate shocks are present, the future price of capital depends on the asset holdings and employment status of each agent. The resultant time-varying distribution of wealth becomes a relevant state variable. We take a dichotomous approach to this problem by initially introducing a generic Walrasian auctioneer who sets the level of aggregate capital and labor in advance for all time periods and all outcomes of the shocks. Conditional on the auctioneer's predictive distribution, households optimize investment-consumption allocations. Given any such market-clearing predictions by the auctioneer—and consequently any prevailing wealth distribution—we derive a *unique* expression (Proposition 1) for the savings function which separates the linear, permanent income component from the nonlinear adjustment arising from incomplete markets and precautionary savings. The auctioneer allows us to assess aggregation even with substantial heterogeneity in agent types and time-varying wealth levels.

Our analysis of aggregation begins with deterministic, two-period economies, Section 3.1. In this setting, we derive conditions under which the combination of incomplete markets and wealth heterogeneity leads to a departure of aggregation even when the wage is deterministic. Proposition 2 delivers a necessary and sufficient condition for the failure of aggregation. Specifically, aggregation fails if a positive measure of households would like to borrow against future labor earnings but are unable to do so due to incomplete markets. Constrained agents can even be "wealthy" in an intertemporal sense (high future earnings), but the market for future earnings is illiquid. Thus, neither incomplete markets nor a skewed wealth distribution alone are sufficient for breaking aggregation.

Our theorems of aggregation (Theorem 1 and Theorem 2) show that adding uncertainty introduces a nonlinear component to the savings function that is well-behaved with respect to household wealth: strictly decreasing, strictly convex, non-negative, with asymptotes that approach zero as household resources increase without bound. Our contribution lies in proving these conditions, which we then exploit to understand aggregation properties. The linear-nonlinear decomposition of the household savings function is preserved but the coefficients governing the rate at which households save change as the horizon lengthens. This, in turn, alters the location of the distribution of wealth, which plays a key role in aggregation. Section 3 concludes with an extension to nominal assets, demonstrating that our approach can be applied to a broader set of models.

Our aggregation theorems are built on a generic treatment of auctioneer and therefore do not rely on any type of simplification. However in order to fully specify an equilibrium, a particular auctioneer is necessary. Section 4 provides an interpretation of the auctioneer as an integral part of numerical algorithms designed to solve dynamic models with heterogeneity. In that setting, the auctioneer breaks the curse of dimensionality by positing a specific aggregate law of motion. Section 4 studies the most popular (numerical) auctioneers and offers the following novel insights:

Section 4.1 shows how to derive an efficient paper-and-pencil analysis of aggregation based on steady-state values that provides a bound on more complex environments. The bound takes advantage of the fact that the steady state capital stock with idiosyncratic uncertainty is strictly larger than the representative-agent (RA) steady state, the primary result of Huggett (1997). Evaluating aggregate wealth at the RA steady-state, we ask the following question: At what point in the RA wealth distribution will households become constrained (would like to borrow against future labor income)? Alternatively, how far in the left-tail is the borrowing constraint binding for households in our economy? We show that this is a trivial calculation and is accurate when the solution is sought in the space of stationary distributions around a steady state.

Section 4.2 studies the popular auctioneer of Krusell and Smith (1998), Castaneda, Díaz-Giménez, and Ríos-Rull (1998), and den Haan (1993) and provides an economic interpretation of the now well-known regression analysis. The intuition of our results date back at least to the optimal portfolio literature of Merton (1969) and Samuelson (1969). With log utility and i.i.d. returns, the optimal portfolio, expressed as a percentage of wealth, is independent of the level of wealth. Technology shocks merely alter the level of wealth in the economy, implying the optimal share of investment is constant and the coefficient of determination (R^2) is identically one under certain assumptions. Thus, we are able to identify regression coefficients and explain the lack of time variation in the aggregate law of motion of capital. We argue that these results do not tie directly to aggregation.

Section 4.3 offers an alternative to the commonly-used metrics of equilibrium determination (e.g., R^2, den Haan (2010)) that exploits the theorems of Section 3 and the dichotomous nature of the problem to produce an arbitrarily precise evaluation of the auctioneer through the market clearing mechanism. First, we demonstrate how the popular metrics like coefficient of determination and those of den Haan (2010) can deliver misleading assessments of equilibrium. One particular counterexample, Example 1, shows that low values for the coefficient of determination do not always imply inaccurate laws of motion. Second, we propose an evaluation step that replaces model-dependent statistics—like the R^2—with an *iterative procedure* for the auctioneer. Therefore, convergence criteria can be applied to the auctioneer's problem in much the same way that it is applied to the household's problem. This is an improvement over contemporary metrics because it does not rely on subjective cutoff points. We test this approach in an economy in which aggregation is not the base case.

1.1 Connection to the Literature

The incomplete-market, heterogeneous-agent model is a workhorse in macroeconomics and we view our household-auctioneer approach as a potential organizing principle. As a specific example, Section 3.4 shows how adding a nominal friction (say, through an intermediate goods sector) does not alter the typical household's consumption / saving decision in a fundamental way, and therefore our insights can be applied to the increasingly important Heterogeneous Agent New Keynesian (HANK) literature [Gornemann, Kuester, and Nakajima (2016), McKay, Nakamura, and Steinsson (2016), Auclert (2019), Kaplan, Moll, and Violante (2018)]. One can then show, through our theorems and decompositions, the connection between the quantitative models of heterogeneity and the alternatives that feature a limited number of agent types [Debortoli and Galí (2018), Bilbiie (2018), Ravn and Sterk (2020), Acharya and Dogra (2020)], as the extent to which an economy aggregates is an obvious way to address questions of comparisons between these models. While Section 3.4 derives the extension to nominal assets, we mainly focus on the seminal Real Business Cycle (RBC) model of Krusell and Smith (1998).

Much of applicable theory on aggregation with incomplete markets dates back several decades and while the results remain useful, they are typically limiting cases of modern models [e.g., Yaari (1976), Constantinides (1982)].[1] However, much of our analysis of the household has precedence in the literature. Given that the savings function is the mirror image of consumption, our treatment of the household dates back at least to the literature examining the permanent income hypothesis [Friedman (1957)]. Deaton (1991) and Carroll (1992) are important partial equilibrium treatments of savings under liquidity constraints. Concavity of the consumption function with respect to wealth under uncertainty was established numerically by Zeldes (1989) and put on firm theoretical foundation by Carroll and Kimball (1996).

More recently Straub (2018) shows that in many common macroeconomic models with precautionary savings motives [Aiyagari (1994), Carroll (1997), Gourinchas and Parker (2002)] have consumption functions that are linear in *permanent income*, which is consistent with the linear component of our savings functions in Theorem 1 and Theorem 2. Carroll (2014) derives a similar limiting condition to the perfect foresight solution as a function of wealth. Bilbiie (2021) employs an assumption of risk-pooling within subfamilies (after idiosyncratic uncertainty is resolved) and assumes family heads that can only partially insure between subfamilies, following Lucas (1990). This serves to facilitate aggregation while preserving some important heterogeneity-related channels. Similarly, Acharya and Dogra (2020) and Ravn and Sterk (2020) make assumptions that mitigate the richness of household heterogeneity along certain dimensions. Specifically, Ravn and Sterk (2020) limit the actions of their three-agent types in order to achieve analytical results with respect to cyclicality of income risk. Acharya and Dogra (2020) show that CARA utility with Gaussian shocks permits linear aggregation with a non-degenerate wealth distribution. While these papers (rightfully) argue that the simplifications which serve to circumvent a potentially infinite-dimensional state space are justified from an economic analysis standpoint, Theorem 1 and

[1] A notable exception is Levine and Zame (2002), who derive conditions in which market incompleteness has little effect on equilibrium allocations, a foretelling result.

Theorem 2 offer an alternative that maintains the richness of the model.

Our contribution to this theoretical literature is twofold: First, we examine the household savings function in general equilibrium through the market-clearing actions of the auctioneer and derive aggregation conditions *endogenously*.[2] One interpretation of the recent literature through the lens of our paper is that invoked assumptions relegate aggregation to exogenously determined parameters. Disaggregation in Bilbiie (2008) posits an exogenous fraction of agents who are hand-to-mouth with key results depending on this fraction; here we look at what determines whether agents will endogenously end up constrained in this manner. Likewise Ravn and Sterk (2020) make simplifying assumptions to place all agents on the constraint, with employed workers sitting right at the point where it becomes binding. Our contribution considers how agents would endogenously end up at this point, with upside income risk being key.

Second, we derive novel *properties* of the nonlinear component (i.e., non-negative, convex, decreasing in wealth, asymptotes) of the household's savings function that are critical for assessing aggregation. Without these asympototic results, we would not be able to bound the savings function under uncertainty, which affords us a clean definition of "approximate aggregation." For a given level of wealth, we can easily calculate the fraction of households that would be constrained and relate this to the nonlinear component. This intuition drives our results in Section 4.1, where we employ our theory to evaluate well-known algorithms and propose extensions thereof.

One of our primary messages—that incomplete markets alone are not sufficient to break aggregation, nor is a skewed distribution of wealth—is not without precedent (in some form) in the literature. For example, Bilbiie (2008) shows that substantial wealth heterogeneity is insufficient to break aggregation, especially when incomes are proportional. Werning (2015) also demonstrates that partial equilibrium analysis is insufficient for assessing aggregation. Despite the fact that we have derived a clean condition for aggregation from the households' perspective, without the auctioneer's values for the interest rate and wage process, our conditions would be indeterminant.

The first generation of models with incomplete-markets (and many contemporary treatments) feature agents who can adequately self-insure with a limited number of assets. While this result has largely been explored numerically, it is robust and first appeared in the asset pricing literature.[3] These papers foreshadowed the "approximate aggregation" results of the heterogeneous agent, macro literature pioneered by Krusell and Smith (1998), whose algorithm seemingly depends on the result. Quoting Krusell and Smith (2006),

[2]Although a different setup and execution, the idea of partitioning the problem of the household and auctioneer dates back at least to Prescott and Mehra (1980). Ljungqvist and Sargent (2004) refer to this as the "Big K (auctioneer), little k (household) trick." Lucas (1994) uses a related "auctioneer algorithm" to clear stock and bond markets in a portfolio problem with heterogeneous investors. Takahashi (2014) is a good example of the importance of relating the household and auctioneer through market clearing.

[3]Following the suggestion of Bewley (1982), several papers examined deviations from the representative-agent framework via incomplete markets in order to explain anomalies such as the Equity Premium Puzzle. The enrichment of the model along this dimension [e.g., Lucas (1994), Aiyagari and Gertler (1991), Heaton and Lucas (1992), Telmer (1993)] was deemed "largely illusionary" by Constantinides and Duffie (1996) because "consumers are able to come close to the complete markets rule of complete risk sharing, even though consumers are allowed to trade in just one security in a frictionless market."

> Approximate aggregation means that aggregates almost do not depend on anything but average capital. The implication of approximate aggregation therefore is that individual decision makers make very small mistakes by ignoring how higher-than-first moments of the wealth distribution influence future prices. If, in contrast, aggregation fails, such moments by definition do influence savings, portfolio decisions, and so on, thus affecting not only the future distribution of wealth, but also average resources available in the future, and hence also future prices relevant to the agent's current decisions. Thus, approximate aggregation allows one to solve the problems of forward-looking agents with a very small set of state variables. This is the key insight. The specific numerical procedure we outline here is the natural one, given this insight.

This algorithm has been widely adopted, including in models that depart substantially from the original KS framework [e.g., Cooley, Marimon, and Quadrini (2004), Zhang (2005), Cooley and Quadrini (2006), Storesletten, Telmer, and Yaron (2007), Khan and Thomas (2013), Favilukis, Ludvigson, and Nieuwerburgh (2017)]. Assessing efficacy of the algorithm requires knowledge of *how* the economy aggregates. The bounded-rational equilibrium could be quite far from the true rational expectations equilibrium. However, our results imply that the algorithm is more robust than the above quote may suggest. Section 4 gives several examples where approximate aggregation fails and yet the optimal share of aggregate capital allocated remains roughly constant. Thus, the law of motion and simulation methods proposed by Krusell and Smith (1998), Castaneda, Díaz-Giménez, and Ríos-Rull (1998) and den Haan (1993) will be an accurate representation of the equilibrium.

Dozens of papers have examined the KS algorithm. The relevant literature is much too voluminous to cite but interested readers are directed to volume 34 of the Journal of Economic Dynamics and Control [den Haan, Judd, and Juillard (2010)], which is devoted to solving heterogeneous agents with incomplete markets and aggregate uncertainty. One paper in this volume of particular relevance is den Haan and Rendahl (2010), who advocate for using aggregation of individual policy rules as a method for finding the aggregate law of motion. While this approach was shown to be an effective method for numerically solving a heterogeneous agent model with aggregate uncertainty, we are advocating for using aggregation properties as an evaluation of accuracy. den Haan and Rendahl (2010) and nearly all applications of the KS algorithm continue to rely on the coefficient of determination, other regression statistics, or non-iterative metrics to test for the equilibrium. Our iterative procedure delivers clear indications of convergence of the KS algorithm as shown in Section 4.3. The methodology is easy to implement and uses the empirical distribution of auctioneer forecasts to substantially improve solution accuracy in cases where the standard coefficient of determination and other well-known statistics prove to be misleading.

2 The Economic Environment

Our results pertain to economic environments that are consistent with the foundational models of this literature [Huggett (1993), Aiyagari (1994), Krusell and Smith (1998)] and by direct extension, their successors. In the main text, we focus primarily on the environment of Krusell and Smith (1998), relegating discussions of generalizations to appendices and footnotes. One notable extension is that of the Heterogenous / Two Agent New Keyesian (TANK/HANK) literature [e.g., Kaplan, Moll, and Violante (2018), Bilbiie (2018), Acharya and Dogra (2020)]. In order to make our connection to this literature more explicit, we include a nominal asset in many of the derivations below.

2.1 Households Households live for T periods indexed by $t = 1, 2, ..., T$.[4] We will use the convention that a new period commences with the arrival of new information. Any variable known or chosen at date t will be indexed by t. Households value consumption according to

$$U(c_1, c_2, ..., c_T) = \mathbb{E}_1 \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{1}$$

where $0 < \beta < 1$ is the intertemporal discount factor and period utility takes the constant relative risk aversion (CRRA) form

$$u(c_t) = \begin{cases} \frac{c_t^{1-\sigma}}{1-\sigma} & \sigma > 0,\ \sigma \neq 1 \\ \log(c_t) & \sigma = 1 \end{cases}$$

Household income in each period is composed of proceeds from a single savings asset and an endowment which is driven by an individual and exogenous stochastic process. Our primary focus will be on production economies in which savings come in the form of capital and endowments in the form of time or efficiency units to devote to labor. As such, we denote the level of savings brought into period $t+1$ by k_t and the endowment in period t by $\ell_t \geq 0$.[5] Savings depreciate at rate $\delta \in [0, 1]$.

Households are price takers. Letting R_t denote the market return on savings net of depreciation and letting W_t denote the price of a unit of endowment in terms of the consumption good, the period resource constraint is given by $c_t + i_t \leq R_t k_{t-1} + W_t \ell_t$, where the left-hand side consists of consumption and investment expenditures at time t and the terms on the right-hand side are savings and endowment income, respectively. We use the convention that lower-case variables are household specific, while upper-case denotes aggregate quantities and prices. An agent's new allocation of capital stock k_t consists of the fraction of current capital which does not depreciate in addition to new investment, $k_t = (1-\delta)k_{t-1} + i_t$. Combining the previous two equations, we

[4] T can be arbitrarily large and vary across households. Results are also given for the limiting case, $T \to \infty$. However, as we note below, control over this parameter will be important for understanding aggregation.

[5] In the production economy of Krusell and Smith (1998), households are endowed with one unit of time and inelastically supply $\epsilon\tilde{\ell}$ units of labor each period, where ϵ is stochastic, taking on the value of unity if employed and zero if unemployed. Hence, $\ell_t \equiv \epsilon_t \tilde{\ell}_t$.

directly write the resource constraint in terms of the household's period t choice of capital holdings,

$$c_t + k_t \leq (1 - \delta + R_t)k_{t-1} + W_t\ell_t \tag{2}$$

Our formal analysis will rely heavily on the intertemporal Euler equation being a necessary condition for optimality in the household's problem. It will therefore facilitate much of the analysis to subject households to their natural borrowing limits $k_t \geq \underline{k}_t$, for $t = 1, ..., T$, in which case the asymptote in the period utility function for zero consumption prevents this constraint from binding. The natural borrowing limit equals the infimum of resource holdings for which the household can guarantee repayment according to its income stream. For example, if the labor endowments are bounded below by zero and there is a positive probability that all future endowments will be null, then the natural borrowing limit is a no-borrowing constraint.

2.2 Firms In the context of a production economy, the savings vehicle available to the households is capital and the endowment comes in the form of labor efficiency units. The income from these assets comes from renting them out to firms which operate in perfectly competitive factor and product markets. The aggregate production technology is Cobb-Douglas, $Y_t = F(Z_t, K_{t-1}, L_t) = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$, with $\alpha \in [0, 1]$. Aggregate capital and labor are denoted K and L respectively, and Z is an aggregate productivity shock. Profit maximization delivers the rental rate of capital and the wage rate as

$$R_t = \alpha Z_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha-1} \tag{3}$$

$$W_t = (1-\alpha) Z_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha} \tag{4}$$

2.3 Uncertainty Let $\mathcal{L}_t$ denote the stochastic processes that enter the model. These shocks can take various forms (e.g., idiosyncratic, aggregate) and can take various correlation structures (e.g., idiosyncratic shocks can be correlated with aggregate shocks). The theoretical results of Section 2.5 and our results on aggregation require only that there be a finite number of possible exogenous outcomes in each future period, which necessitates shocks with discrete sample spaces. This allows us to write theorems with expectations as finite sums, which we then algebraically manipulate. While this restriction does rule out commonly used continuous support processes, for example autoregressive series with normal innovations, it admits finite approximations to such series commonly used in numerical solutions.

Assumption 1: For each period $t = 2, ..., T$ of a given household's lifetime, there is a finite set $\mathcal{S}_t \subset \mathbb{R}^2$ such that exogenous shock distributions assign probability one to the event $(\ell_t, Z_t) \in \mathcal{S}_t$.

2.4 The Auctioneer Households maximize their preferences subject to the above budget constraint and borrowing limit, given initial savings k_0. To do so, they must have access to sequences of state contingent prices, R_t and W_t, for their entire lifetime. The equilibrium sequences of prices

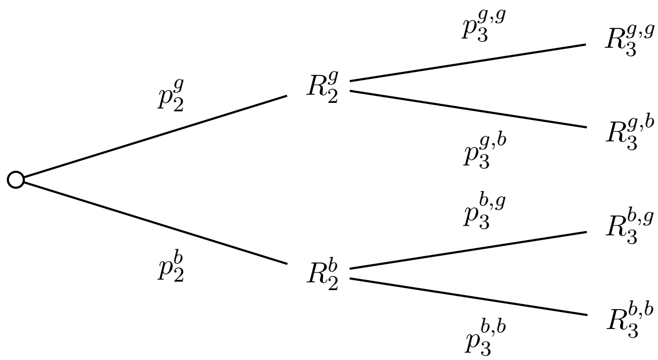

Figure 1: Example of auctioneer predictions for $T = 3$. We assume households know all possible future realizations of the interest rate (R), and the states (good g, bad b) in which they occur. Households are endowed with knowledge about the future probabilities of each such state (p). Subscripts denote time periods and superscripts states of the world.

will clear markets in all future states, and consequently depend upon the evolution of aggregate capital and labor. These, in turn, depend upon the time-varying, cross-sectional distribution of individual capital holdings. That this distribution is a relevant state variable and (potentially) a high-dimensional object is the primary challenge of these models.

To establish our aggregation results in the most generic environment possible, we introduce a Walrasian auctioneer whose job is to report to the households *all* possible future prices which could occur during their lifetimes and the states in which they occur. Formally, we will denote by $\mathcal{F}_t$ the predicted future prices relevant to the consumption / savings decision of an age t household,

$$\mathcal{F}_t = \{\boldsymbol{R}_s, \boldsymbol{W}_s\}_{s=t+1}^{T} \tag{5}$$

These predictions associate to each future age s a vector (indicated by bold font) of prices, with one component for each possible history of nature when the household reaches this age, given the current state at age t. Because households are price takers, they treat these predictions as given when forming expectations and optimizing. Thus, there is a natural dichotomy between the auctioneer's and households' problems.

To visualize the role of the auctioneer, it is helpful to think of the tree diagram shown in Figure 1 for the case $T = 3$ and for two possible aggregate states, good (g) and bad (b). The auctioneer provides the households with predictions R_t^j for the interest rate at each time in every possible future. As we will formalize in the next section, households endowed with knowledge of the corresponding probabilities of each future can then optimize their consumption and savings. The auctioneer may then aggregate these decisions in each possible future and compare the implied prices with her predictions to determine whether markets will clear.

Our theorems for the household are built on a *generic* treatment of these predictive densities and therefore do not rely on any type of simplification. From the perspective of the household, the predictive distributions can be arbitrarily large and accurately reflect the true rational expectations equilibrium. In this respect, our aggregation results are immune from the curse of dimensionality.

Moreover, the generality of our aggregation theorems allows us to study various types of auctioneers that are an integral part of popular numerical algorithms designed to solve models with heterogeneity.

2.5 Formal Treatment of a Household's Problem Along with the auctioneer's predictions, the household is assumed to know the joint distribution of the idiosyncratic and aggregate exogenous shock processes, $\mathcal{L}_t = \{\boldsymbol{\ell}_s, \boldsymbol{Z}_s, \boldsymbol{p}_{\ell,\mathrm{z},s}\}_{s=t+1}^{T}$, consistent with rational expectations. We can now write the household's problem formally as follows: Letting x_t denote period resources at time t, $x_t = (1-\delta+R_t)k_{t-1} + W_t\ell_t$, the dynamic programming formulation consists of the sequence of problems

$$\begin{aligned} V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \quad &= \quad \max_{c_t, k_t} \Big(u(c) + \beta \mathbb{E}_t V^{(t+1)} \left(x_{t+1}, \mathcal{L}_{t+1}, \mathcal{F}_{t+1} \right) \Big) \\ \text{subject to} \qquad & \qquad c_t + k_t \le x_t \\ & \qquad x_{t+1} = (1-\delta+R_{t+1})k_t + W_{t+1}\ell_{t+1} \\ & \qquad k_t \ge \underline{k}_t \end{aligned} \tag{6}$$

for $t = 1, ..., T$, with k_0, ℓ_1, R_1 and W_1 given, along with the terminal condition $V^{(T+1)} \equiv 0$.

A solution to this sequence indicates savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$, giving a typical household's choice of additional asset holdings as a function of current resources and predictions, and this will be our primary object of focus.[6] It will therefore be important that the households' problems have a unique, well-behaved solution, as our first proposition establishes.

Proposition 1: Household Existence and Uniqueness. There is a unique solution to the household's dynamic programming problem (6). The associated savings functions $k^{(t)}$ are increasing (strictly for $t < T$) with respect to x_t and satisfy $\lim_{x_t \to \underline{k}_t} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \underline{k}_t$, $\lim_{x_t \to \infty} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \infty$, for $t < T$. The corresponding value functions are strictly increasing and strictly concave with respect to x_t and satisfy $\lim_{x_t \to \underline{k}_t} V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = -\infty$

Proof. Appendix A. □

Our assumptions ensure that the inter-temporal Euler equations are a necessary condition for optimality. In terms of the asset choice, these conditions read

$$(x_t - k_t)^{-\sigma} = \beta \mathbb{E}_t (1-\delta+R_t)(x_{t+1} - k_{t+1})^{-\sigma}, \quad t = 1, ..., T-2 \tag{7}$$

$$(x_{T-1} - k_{T-1})^{-\sigma} = \beta \mathbb{E}_t (1-\delta+R_T) x_T^{-\sigma} \tag{8}$$

The theory below relies on algebraic manipulations of these expressions to study aggregation. The omnipresent nature of the Euler equation implies that our theorems are operational in many

[6]We use the notation $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$ to denote a function with dependence on the elements $x_t, \mathcal{L}_t, \mathcal{F}_t$. This notation is used when referring to the macroscopic properties of the savings function (i.e., across different wealth levels). In what follows, we often drop the dependence on the predictive distributions $\mathcal{L}_t, \mathcal{F}_t$ for notational convenience.

environments. While our focus is on the neoclassical growth model with idiosyncratic labor shocks, our approach can be adapted to several other popular model choices. As we note below and derive explicitly in the online Appendix, introducing nominal assets does not alter the household's problem substantially. All of our theorems continue to go through with relatively minor algebraic adjustments.

3 Aggregation Theorems

This section builds sequentially to an understanding of nonlinearity in household savings and how it impacts aggregation. We start from a two-period model, which permits closed-form expressions of the equilibrium, and demonstrate that the intuition derived therein extends to more complex environments.

3.1 Two-Periods, No Uncertainty Consider a typical household's problem with a single savings decision ($T = 2$), log utility ($\sigma = 1$), and complete depreciation of capital ($\delta = 1$). Further, suppose there is no uncertainty about labor income or aggregate productivity. Markets are complete in that there exists a spot market at $t = 1$ to trade claims on future wages. The auctioneer provides deterministic values for the aggregate wage (W) and interest rate (R). The typical household's optimal savings function (capital holding, k_1) for $T = 2$ is

$$k_1 = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2 \ell_2}{R_2}\right) \tag{9}$$

where x_1 is defined as household wealth known at the beginning of period 1. Savings that optimally smooths consumption across periods compares wealth today with discounted labor income tomorrow. If wealth is relatively high, savings will be positive. If wealth is significantly less than discounted wages, the household will borrow against future labor earnings to bring resources into the current period.

The aggregation properties of this model are also straightforward. The household's savings function is linear in wealth with slope coefficient $\beta/(1 + \beta)$, which is the rate of transformation of wealth. Since all households, regardless of wealth levels, have the same marginal propensity to save (consume), the economy Gorman (1953, 1961) aggregates. We will refer to savings functions with this characteristic as linear in wealth. Integrating over all households gives the well-known representative agent aggregate savings function,

$$K_1 = \left(\frac{\alpha\beta}{1+\alpha\beta}\right) X_1 \tag{10}$$

where X_1 is economy-wide resources available at $t = 1$. Note that this value of aggregate capital is what the auctioneer would provide to households and is independent of the distribution of wealth.

Suppose now that markets are incomplete and households cannot borrow against future labor income. If we impose an ad-hoc no-borrowing constraint ($k_1 \geq 0$), the optimal savings function is piece-wise linear

$$k_1 = \begin{cases} \left(\frac{\beta}{1+\beta}\right) x_1 - \frac{W_2 \ell_2}{(1+\beta)R_2}, & \text{if } x_1 \geq \frac{W_2 \ell_2}{\beta R_2} \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Now households which would borrow under complete markets can no longer intertemporally substitute. These constrained households behave in a hand-to-mouth fashion, consuming all of the resource endowment in the first period and labor income in the second. We will refer to the households which choose a positive level of savings as unconstrained, denoted by "**u**" and will denote households whose constraint binds with "**c**."

We take as given an initial distribution of household resources, λ_1. (Foreshadowing results, one may think of this distribution of wealth as being determined by a simulated economy close to its steady state distribution.) If this distribution is such that every household is unconstrained, we have

$$x_1 \geq \frac{W_2 \ell_2}{\beta R_2} = \frac{(1-\alpha) Y_2 \ell_2 K_1}{\alpha \beta Y_2 L_2} = \left(\frac{1-\alpha}{1+\alpha\beta}\right) \frac{\ell_2}{L_2} X_1$$

for almost every x_1 in the support of λ_1. This level of savings is linear in initial resources for all households and aggregation once again obtains, with aggregate savings being given by (10). Rearranging slightly, we obtain a necessary condition for the economy to aggregate,

$$\frac{x_1}{X_1} \geq \left(\frac{1-\alpha}{1+\alpha\beta}\right) \frac{\ell_2}{L_2} \tag{12}$$

Namely, (almost) every household must have a share of initial resources which is sufficiently large relative to its share of the aggregate wage bill. We can use (12) to obtain a necessary condition[7] for the failure of aggregation, which we state as a proposition:

Proposition 2: Skewness, Labor Market Outcomes and Aggregation. A necessary condition for breaking aggregation is for the initial resources to be sufficiently skewed relative to labor market outcomes:

$$\frac{\int_{\mathbf{c}} \frac{x_1}{X_1}\, d\lambda_1}{\int_{\mathbf{u}} \frac{x_1}{X_1}\, d\lambda_1} < \frac{\int_{\mathbf{c}} \frac{\ell_2}{L_2}\, d\lambda_1}{\frac{\alpha(1+\beta)}{1-\alpha} + \int_{\mathbf{u}} \frac{\ell_2}{L_2}\, d\lambda_1} \tag{13}$$

Incomplete markets alone are not sufficient to break aggregation, *nor is a skewed distribution of wealth*. There must be a positive measure of households that would like to borrow against

[7] Sufficiency attains if and only if there exists a set of pairs of initial resources and final labor outcomes, $\mathbf{c} = \{(x, \ell)\}$, such that $\lambda_1(\mathbf{c}) > 0$ and such that for every (x, ℓ) in the set of constrained allocations, we have

$$\frac{x}{\int_{\mathbf{u}} x_1\, d\lambda_1} < \frac{\frac{\ell}{L_2}}{\frac{\alpha(1+\beta)}{1-\alpha} + \int_{\mathbf{u}} \frac{\ell_2}{L} d\lambda_1}$$

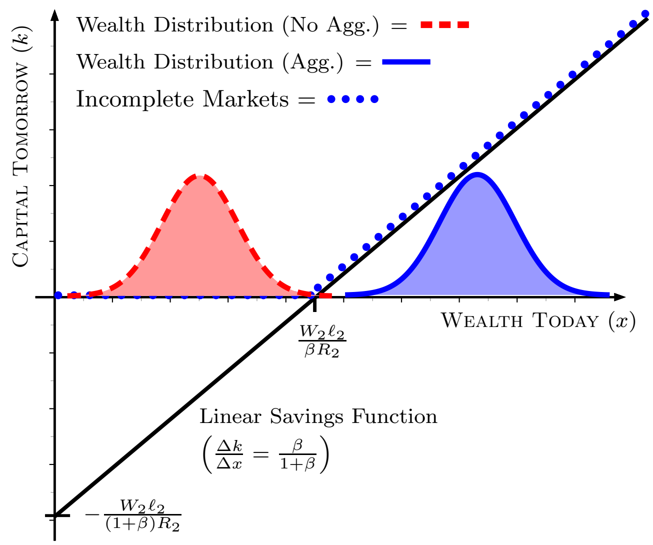

Figure 2: Savings Functions in k-x Space. The Linear Savings Function (black line) and the Incomplete-Markets Savings Function (dotted line) are plotted for various wealth distributions (λ_1). Two distributions of wealth are shown; one that leads to aggregation (solid, blue distribution) and one that does not (dashed, red distribution).

future labor earnings that are unable to do so. Aggregation would continue to hold if relatively poor households had bleak labor market prospects. Conversely, aggregation would fail to hold if relatively wealthy households expected much higher wages in the future. The distribution of wealth alone does *not* determine aggregation properties. The denominator on the RHS of (13) shows that the failure of aggregation requires a more skewed wealth distribution as labor's share decreases, patience increases, or the fraction of labor hours supplied by the poor decreases. Moreover, allowing for partial depreciation of capital ($\delta \in (0, 1)$) would facilitate aggregation by increasing the return to savings, requiring a stronger inequality in (13).

Figure 2 plots the savings functions k_1 against initial resources x_1 under incomplete markets (dotted line) and complete markets (solid line). With complete markets, households can fully insure across states of nature via state-contingent assets. They can borrow up to tomorrow's discounted labor earnings (k-intercept of linear savings function) and savings turn negative when the household's wealth (x_1) falls below tomorrow's discounted wage, (x intercept). Under linear savings and for any point on the distribution of wealth, the slope of the savings function is constant at $\beta/(1 + \beta)$. Every households' marginal propensity to consume is identical. Aggregation holds independent of the distribution of wealth (λ_1).

When we do not allow households to borrow (dotted line), households save the same constant fraction until wealth falls below the discounted wage. At that point, savings equal zero. The extent to which the economy aggregates depends upon the relationship between the distribution of wealth and labor market outcomes. For heuristic purposes, we plot two different distributions in Figure 2 to represent two different economies, acknowledging that the numeric values of the x-intercept would not be identical across the economies. As shown above, if no households are constrained $\lambda(\mathbf{c}) = 0$ (solid, blue distribution), then aggregation holds. If the distribution of wealth falls below the intercept of the Gorman savings function (dashed, red distribution), then households would be constrained $\lambda(\mathbf{c}) > 0$ and the economy will not aggregate. It is the relationship between labor market outcomes as determined by the piece-wise linear function and the distribution of wealth that determines aggregation.

As we justify more thoroughly below, this simple model provides the basic intuition for aggregation in dynamic economies. Our next results will show that the entire class of models described thus far share the feature that adding risk to the economy results in a well-behaved perturbation to the closed form solution of (9). This perturbation will be seen to reflect household risk aversion under uncertainty.

3.2 Two-Periods, Uncertainty

We now introduce idiosyncratic labor shocks to the two-period setup described above. We assume households face a "low" labor outcome which we take to be an unemployment state, $\ell_{\text{low}} = 0$, with probability p, and a "high" labor outcome, ℓ_{high}, with probability $1-p$. Capital must now serve the dual role of the savings vehicle to intertemporally smooth consumption and as insurance against employment shocks. This, combined with the risk-averse nature of our households, will result in savings functions which are nonlinear across the entire feasible set of resources. Hence the savings decisions will not aggregate for non-degenerate distributions of households across resources. Our results are summarized by the following theorem.

Theorem 1: Two Periods, CRRA Utility. The savings function for the typical household with wealth level x_1 and CRRA utility can be written as

$$k^{(1)}(x_1) = \frac{1}{1+Q_1}\left(Q_1 x_1 - \mathbb{E}\left(\frac{W_2\ell_2}{1-\delta+R_2}\right) + \epsilon^{(1)}(x_1)\right) \tag{14}$$
$$\text{with} \quad Q_1 = \left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}$$

where the nonlinear error term is strictly decreasing, strictly convex, non-negative, and satisfies

$$\lim_{x_1\to\underline{k}_1} \epsilon^{(1)}(x_1) = \mathbb{E}\left(\frac{W_2\ell_2}{1-\delta+R_2}\right) + \underline{k}_1, \qquad \lim_{x_1\to\infty} \epsilon^{(1)}(x_1) = 0$$

Proof. See Appendix A.

An obvious corollary of the theorem is the log utility case, $\sigma = 1$, which gives the savings function as

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \mathbb{E}_1\left(\frac{W_2\ell_2}{1-\delta+R_2}\right) + \epsilon^{(1)}(x_1)\right) \tag{15}$$

where the nonlinear error term $\epsilon^{(1)}(x_1)$ satisfies the properties of Theorem 1.

The first two terms on the right-hand side of the household's savings equation (14) and (15) are

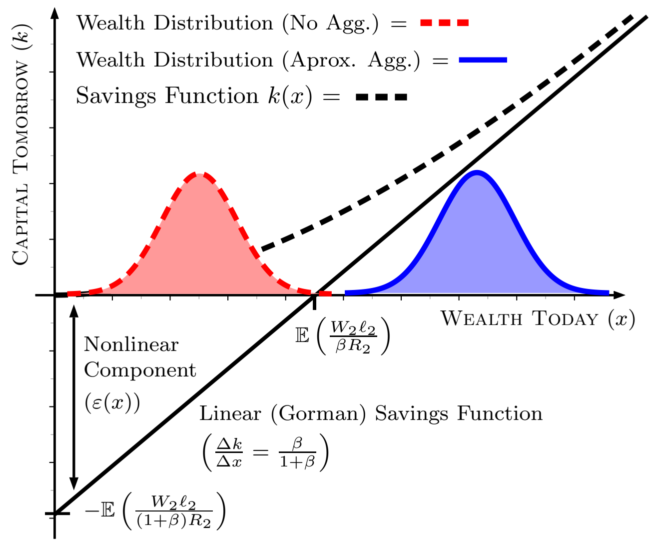

Figure 3: Savings Functions in k-x Space. The Linear (Gorman) Savings Function (black line) and the household's Savings Function (dashed line) are plotted for various wealth levels. Two initial distributions of wealth are shown; one that leads to approximate aggregation (solid, blue distribution) and one in which aggregation will not hold (dashed, red distribution).

the same as the deterministic case (9), with the deterministic wage replaced by the expected wage.[8] Households once again compare current wealth levels to (expected) labor market outcomes. The final term, $\epsilon^{(1)}(x_1)$, is unique to the uncertainty case, strictly positive and captures the precautionary savings motive of the household. Thus, the savings function is nonlinear in wealth and aggregation will not hold in general. The extent to which the economy aggregates depends upon the size of this last term. We therefore refer to it as the nonlinear component in the otherwise linear (Gorman) savings rule. If this nonlinear component is small, the economy will "approximately aggregate." If it is large, aggregation breaks down.

The nonlinear term has the disadvantage that it is endogenous, with a closed-form expression that is inaccessible to us. Nonetheless, we can derive properties of this nonlinear component from what we know about the savings function. For example, differentiating $\epsilon^{(1)}(x_1)$ and recalling that $k^{(1)}$ is strictly increasing, we obtain that the nonlinear component is strictly decreasing in resources. We can also show that it is positive, strictly convex in wealth, and we can calculate its limits as resources approach the endpoints of the domain. The first limiting case shows that as resources dwindle, the savings function approaches the natural borrowing limit, $\underline{k}_1$. If unemployment occurs with positive possibility and there is no unemployment benefit, this value will be zero. As wealth increases, the value of the nonlinear component diminishes, and the household will behave as a linear (Gorman) saver.

[8]The difference between the utility specifications can be seen in the linear (Gorman) savings function, which has a slope of $Q_1/(1+Q_1)$, that differs from the log case of $\beta/(1+\beta)$ due to income/substitution effects. In addition, risk aversion plays an important role in the nonlinear component. We examine these elements more closely in the next section.

Figure 3 adds the nonlinear savings function $k(x)$ (dashed line) to Figure 2. Theorem 1 establishes that the savings function limits to zero as wealth falls to the natural borrowing limit, and asymptotes to the linear (Gorman) savings function as wealth increases. The nonlinear component is, therefore, decreasing in wealth. We plot this figure assuming log utility, but replacing β with Q as defined in (14) and increasing the size of the nonlinear component delivers the CRRA savings function.

As in the deterministic case, the relationship between expected labor market outcomes and the distribution of wealth is critical. If the distribution of wealth lies primarily to the left of the discounted, expected wage (x-axis intercept), aggregation will not attain. This is depicted by the dashed, red distribution in Figure 3. If instead, the wealth of most households is substantially greater than the expected wage (solid, blue distribution), the nonlinear component of the savings function will be minimal, and the economy will approximately aggregate. The "approximate" adjective applies because even if the poorer households have significant nonlinear savings behavior, they hold such little capital that their impact on aggregates is minimal.

Our linear-nonlinear decomposition provides some intuition as to how two-asset models or models with illiquid assets might change our calculus. As an example, suppose we have a large share of "wealthy hand to mouth" with illiquid asset holdings. Illiquidity can be interpreted as a missing market and by imposing borrowing limits, we have enforced a form of illiquidity. When markets are incomplete, households would like to borrow but cannot. Likewise with illiquid assets, households cannot transfer these resources directly into consumption-equivalent goods. Our interpretation also allows for "wealthy hand to mouth" as households with high levels of wealth that are constrained by incomplete markets *and* have increasingly improving labor outcomes, would have high marginal propensities to consume. This scenario would lead to a breakdown of aggregation despite a high-level of wealth. As we note above, incomplete markets alone are not sufficient to break aggregation, neither is a skewed distribution of wealth. There must be a positive measure of households that would like to borrow against future labor earnings that are unable to do so. Illiquidity is one way to impose this condition.

3.3 T-Periods, Uncertainty

We now present the extension of Theorem 1 to the full generality of the setup. While the theorem shows that the properties derived in the two-period model extend to the multi-period counterpart for CRRA (and log) utility, the parameter values that influence aggregation change in important ways.

Theorem 2: Typical Household Savings Function, $\sigma \neq 1$. Make the sequence of recursive definitions, $M_T = (1-\delta+R_T)^{1-\sigma}$; $Q_{T-1} = (\beta \mathbb{E}_{T-1} M_T)^{1/\sigma}$; $M_t = (1-\delta+R_t)^{1-\sigma}(1+Q_{t+1})^{\sigma}$; $Q_{t-1} = [\beta \mathbb{E}_{t-1} M_t]^{1/\sigma}$ for $t = T-2, ..., 1$. The savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ which solve the household's optimization problem can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{Q_t}{1+Q_t} x_t - \frac{1}{1+Q_t} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \tag{16}$$

where the nonlinear error term $\epsilon^{(t)}$ is strictly decreasing, strictly convex, non-negative, and satisfies

$$\lim_{x_t \to \underline{k}_t} \epsilon^{(t)}(x_t) = \frac{1}{1+Q_t} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \underline{k}_t \right]$$

and

$$\lim_{x_t \to \infty} \epsilon^{(t)}(x_t) = 0$$

Proof. Appendix A. □

The form of the savings function (16) is identical to the two-period counterpart in that the linear component compares current wealth x_t, discounted at a rate that is constant across all households, to expected discounted labor earnings. Additionally, the error term satisfies the same properties as the two-period model (i.e., non-negative, convex, decreasing in wealth, identical bounds given by asymptotes).

The difference is in the *values* of the parameters as T increases. Consider the limiting case of Theorem 2 as $T \to \infty$ with log utility, which we state as a conjecture.[9]

Conjecture 1: Infinite Horizon, Log Utility. The savings functions $k^{(t)}(x, \mathcal{L}_t, \mathcal{F}_t)$ which solve the infinite horizon analogue with log utility can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \beta x_t - (1-\beta) \mathbb{E}_t \left(\sum_{s=t+1}^{\infty} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \tag{17}$$

where the nonlinear error term $\epsilon^{(t)}$ is strictly decreasing, strictly convex, non-negative and satisfies the well-established limits.

The horizon T is a critical parameter in our analysis of aggregation. As the horizon increases, consumption smoothing necessitates that households save an increasing fraction of their wealth,

[9] The only substantial difference here is that the household now must be endowed with predictive probabilities into the infinite future and standard transversality conditions replace zero savings at T. Naturally, for the conjecture to make sense, the limits which define the sums and products out into the infinite future must be well defined, which in turn requires that household predictions must be restricted to be well-behaved in an appropriate sense. Putting aside this difficulty, the strategy for proving this seems evident. If we consider Euler equation iteration to be an operator on an appropriate function space, our main theorems essentially state that this operator leaves invariant a particular subspace. If this subspace is closed in some topology in which repeated application of the operator leads to convergence, then the above conjecture will hold.

which is well known permanent income behavior. This can be seen by comparing (17) to the two-period analog (15). The fraction of wealth saved increases substantially from $\beta/(1+\beta)$ in the two-period model to β as $T \to \infty$. Moreover, the subjective discount applied to both the expected stream of wages and the nonlinear component approaches $(1-\beta)$ as $T \to \infty$. A sufficiently high discount factor (the typical quarterly calibration ranges from 0.95 to 0.99) places a much higher weight on savings relative to the expected wage and the nonlinear component of the savings function, facilitating aggregation. Households continue to compare expected wages to wealth but the discount applied to wages is significantly lower for substantial T.

As a graphical representation, imagine adding the savings function (17) to Figure 3. As T increases so does the slope of the linear (Gorman) savings component; the x- and y-intercept would move closer to the origin, ceteris paribus. The linear savings function for $T \to \infty$ would lie everywhere to the left of the linear savings function for $T = 2$. For a fixed distribution of wealth, the bounds on the savings function imply a smaller nonlinear component as T increases and aggregation would be much more likely to hold.

3.4 Aggregation with Nominal Assets

We now introduce nominal assets (and other considerations consistent with the HANK setup) in order to provide intuition as to how it would alter aggregation. Consider the following two-period budget constraint,

$$P_1 c_1 + b_1 = I_1 b_0 + W_1(1-\tau_1)z_1 + P_1 d_1 + P_1 T_1$$
$$P_2 c_2 = I_2 b_1 + W_2(1-\tau_2)z_2 + P_2 d_2 + P_2 T_2$$

where P_t denotes the time t price level, I_t the time t gross nominal rate, τ_t a time t labor income tax rate, d_t profits from monopolistic firms, and T_t a lump sum government transfer. As has been established, aggregation depends upon the linearity of the savings / consumption function. If profits (d_t) and government transfers (T_t) are lump sum, then their impact on aggregation will be negligible. Moreover, we can embed the uncertainty associated with distorted tax rates directly into the exogenous shocks, letting $\tilde{z}_t = (1-\tau_t)z_t$, with no change to our formulas.[10] Our interest lies in how the price level and nominal interest rate impact our linear-nonlinear decomposition to which we have the following corollary to Theorem 1 for log utility.[11]

Corollary 1: Two Periods, Log Utility, Nominal Assets. The savings function for the typical household with wealth level x_1 and log utility can be written as

$$\frac{b_1}{P_1} = \left(\frac{\beta}{1+\beta}\right)x_1 - \frac{1}{1+\beta}\mathbb{E}\left[\frac{\mathcal{W}_2}{R_2}z_2\right] + \left(\frac{1}{1+\beta}\right)\frac{\mathrm{Var}(\zeta_2)}{\frac{b_1}{P_1} + \frac{\mathcal{W}_2}{R_2}z_{\mathrm{low}} + \frac{\mathrm{Var}(\zeta_2)}{\mathbb{E}\zeta_2}} \tag{18}$$

[10] Our setup does not allow for elastic labor supply, which is not an innocuous alteration as the Euler equation—from which we derive the linear-plus-error form—contains wage earnings, hence hours. Hours worked are an implicit nonlinear function of savings, and therefore introduce additional nonlinearities to this part of the savings function. However, our asymptote remains intact as these nonlinearities approach zero in the limit of high wealth.

[11] We only examine log utility and two periods as we have established that the intuition carries over to multiple periods and CRRA utility with explicit derivations available upon request.

with $b_1 \geq \underline{b}_1$ and where $\Pi_2 = P_2/P_1$ is inflation, $R_2 = \Pi_2/I_2$ is the real rate, $\mathcal{W}_2$ is the real wage, and $\zeta_2 = \frac{\Pi_2 W_2}{I_2}(z_2 - z_{\text{low}})$ denotes real excess wages.

Proof. See Appendix A.

The extent to which nominal assets impact aggregation depends upon the stickiness of wages. If wages are perfectly flexible, then an increase in inflation will be offset in all of the wage terms by an increase in the nominal wage, leaving the real wage and the linear component of savings unaffected. To see this note,

$$\frac{\partial(b_1/P_1)}{\partial P_2} = -\left(\frac{1}{1+\beta}\right)\frac{\text{Var}(\zeta_2)}{\left(\frac{b_1}{P_1} + \frac{\Pi_2\mathcal{W}_2}{I_2}z_{\text{low}} + \frac{\text{Var}(\zeta_2)}{\mathbb{E}\zeta_2}\right)^2}\frac{\partial((b_1/P_1)}{\partial P_2} \tag{19}$$

which requires $\partial((b_1/P_1)/\partial\Pi_2 = 0$, so the real value of savings does not respond to the change in prices/inflation. However, if wages are sticky (a common assumption in the New Keynesian literature), then an increase in the price level will not be offset by changes in the real wage. Revisiting Proposition 2, a falling expected real wage will facilitate aggregation in the sense that households would like to borrow less today due to deteriorating labor prospects. However, inflation uncertainty increases the precautionary savings motives, especially as the variance of inflation and spread of the distribution increases. The extent to these second-moment concerns outpace the first-moments of inflation obviously depends upon the stochastic process for inflation. For example, as US inflation data has become less volatile over the last two decades, the precautionary savings motives attributable to inflation have most likely become negligible.

Also note that $\text{Var}(\zeta_2)$ scales up with the real discounted wage and with underlying income risk itself. The literature has emphasized the importance of cyclicality of earnings risk [Bilbiie (2021), Ravn and Sterk (2020), Acharya and Dogra (2020)]. Procyclical income risk exacerbates $\text{Var}(\zeta_2)$, implying higher precautionary savings.

4 Implications

The aggregation theorems of the previous section are proved for a generic auctioneer. This generality affords us the opportunity to study several different types of auctioneers that are popular in the literature.[12] We show how aggregation can be quickly assessed using the auctioneer proposed by Aiyagari (1994) and Huggett (1997). We then study the auctioneer of Krusell and Smith (1998), Castaneda, Díaz-Giménez, and Ríos-Rull (1998) and den Haan (1993), which has served as the most popular numerical approach to solving macroeconomic models with heterogeneity. We offer a novel interpretation of the auctioneer's output and an improvement on the numerical procedure.

[12] As such, our focus will be primarily on dynamic economies. In two-period environments, the modeler has complete control of the initial distribution of wealth and can make households sufficiently poor (wealthy) to break (preserve) aggregation. While controlling the extent to which an economy aggregates can be useful for many thought experiments [Chatterjee (1994)], it is counter to standard dynamic analysis which seeks to eliminate dependence on initial conditions. Nearly all dynamic heterogeneous agents models study the *evolution* of the distribution of wealth and/or examine perturbations around some well-defined steady state, and we follow suit.

4.1 Steady-State Auctioneer and a Paper-Pencil Assessment of Aggregation Consider the steady-state auctioneer, which specifies a single, constant value for all prices in all periods. In a pure credit setting, this amounts to a Huggett Auctioneer which fixes a constant interest rate [Huggett (1993)]. In a production setting, this can take the form of an Aiyagari Auctioneer, which specifies a single aggregate capital labor ratio K/L for every future period and computes prices consistent with firm optimization, equations (3) and (4) [Aiyagari (1994)].

Steady state analysis is often used to simplify the economics and provide insights that are of the paper-and-pencil variety. We show how this carries over to the analysis of aggregation with the help of a steady-state auctioneer. Consider substituting steady-state auctioneer values, $\mathcal{F}_{ss} = \{K_{ss}, L_{ss}, R_{ss}, W_{ss}\}$, into the typical household's savings function (17) under log utility and for $T \to \infty$. Because we do not have a closed-form expression for the nonlinear component, $\epsilon^{(t)}$, we cannot solve directly for the corresponding steady state values.[13] However, we can use the theorems of the previous section and the main results of Aiyagari (1994) and Huggett (1997) to provide an assessment of aggregation that only requires a few lines of algebra.

Huggett's primary result is that when idiosyncratic uncertainty is present, the steady state capital stock obeys the inequality $\beta f'(K) < 1$ (Theorem 1, pg. 391), where production follows $f(K) \equiv F(K,1) + (1-\delta)K$ and $F(\cdot)$ satisfies the usual constant returns to scale assumption. Thus, the steady state capital stock with idiosyncratic uncertainty is strictly larger than the representative-agent (RA) steady state, K_{RA}. If we fix the amount of capital in the heterogeneous-agent economy to the RA steady state value, knowing that the true value is higher due to the result of Huggett (1997), we can then ask: at this level of wealth, what fraction of households are constrained, $\lambda(\mathbf{c})$? If this value is low (high), the economy will (not) approximately aggregate.

To provide a more concrete example, consider the following thought experiment: a lazy economist wants to understanding the approximate aggregation result of Krusell and Smith (1998) without having to do any numerical analysis (hence the "lazy" moniker). The economist has on hand the typical household's savings function under log utility, (17), and recalls that the unique RA steady state is given by

$$\frac{K_{\text{RA}}}{L_{\text{RA}}} = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}, \qquad W_{\text{RA}} = (1-\alpha)\left(\frac{K_{\text{RA}}}{L_{\text{RA}}}\right)^{\alpha}, \qquad R_{\text{RA}} = \beta^{-1} - (1-\delta)$$
$$Z_{RA} = 1, \;\; \tilde{Y}_{\text{RA}} = K_{\text{RA}}^{\alpha} L_{\text{RA}}^{1-\alpha} + (1-\delta)K_{\text{RA}} = L_{\text{RA}}\left(\left(\frac{K_{\text{RA}}}{L_{\text{RA}}}\right)^{\alpha} + (1-\delta)\frac{K_{\text{RA}}}{L_{\text{RA}}}\right)$$

Substituting these steady state values into the savings function (17) and assuming[14] that, on average, households are employed at constant rate π_e gives the typical household's savings function

[13] In a continuous-time setup, Achdou, Han, Lasry, Lions, and Moll (2021) show that there is a unique stationary equilibrium if the intertemporal elasticity of substitution is greater than one.

[14] Here we follow the literature in assuming that the endowment is transformed to labor input according to $\ell_t = \bar{\ell} e_t$, where $\bar{\ell}$ is a normalizing constant and e_t is the employment shock unique to each household. Note that our approach can accommodate a more sophisticated process for the idiosyncratic labor shock.

as

$$k^{(t)} = \beta\left(\left(\frac{K_{\mathrm{RA}}}{L_{\mathrm{RA}}}\right)^{\alpha} + (1-\delta)\frac{K_{\mathrm{RA}}}{L_{\mathrm{RA}}}\right)\omega_t - \beta(1-\alpha)\left(\frac{K_{\mathrm{RA}}}{L_{\mathrm{RA}}}\right)^{\alpha}\pi_e + \epsilon^{(t)} \tag{20}$$

where household wealth, x_t, is denoted as a share ω_t of total resources $\tilde{Y}_{\mathrm{RA}}$. The economist can then solve for the share of wealth which represents the point in the distribution where households would like to borrow against future labor earnings in the linear (Gorman) savings function. This value is found by setting ϵ_t and (20) equal to zero, and solving for ω. This particular share of wealth is the x-intercept in the linear savings function in Figures 2 and 3, and is given by

$$\omega^{\star}\,|_{\mathrm{RA}} = \left(\frac{(1-\alpha)\pi_e}{1+\alpha(1-\delta)/R_{\mathrm{RA}}}\right) = \left(\frac{(1-\alpha)\pi_e(1-\beta(1-\delta))}{1-\beta(1-\delta)(1-\alpha)}\right) \tag{21}$$

The value of wealth given by (21) represents a point of substantial curvature in the household's savings function. The intuition follows from our main theorems in Section 3, which prove that the savings function would be driven nearly entirely by the nonlinear component $\epsilon^{(t)}$ when evaluated at $\omega^{\star}$. This intuition also accords with Huggett (1997), who shows that the inequality $\beta f'(K) < 1$ can neither be attributed to a high rate of time preference relative to the interest rate nor a positive third derivative of the period utility function, but is due to binding borrowing constraints. Our economist plugs in the quarterly calibration of Krusell and Smith (1998) ($\alpha = 0.36, \beta = 0.99, \delta = 0.025$), coupled with the assumption that agents have a 4% chance of unemployment each period, to get $\omega^{\star} = 5.6\%$. In this case, the borrowing constraint binds at only 5.6% of steady-state wealth (capital holdings of slightly greater than 2 given a RA steady state capital stock of roughly 38), suggesting that very few households will have nonlinear savings functions.

How does our economist know that this metric is a reliable statistic? Accuracy of this measure is lost if the RA steady state does not have substantial point mass in the heterogeneous agent distribution. However, this is not likely given the focus on stationary distributions that has been adopted by (nearly) the entire literature [Krusell and Smith (2006)]. Many of the best numerical approaches employed to solve models with heterogeneity rely on grids and perturbation methods that are built around steady states [e.g., Reiter (2010), Winberry (2018), Boppart, Krusell, and Mitman (2018)]. Indeed, plotting a vertical line at the RA steady state capital stock would be a value close to the mode of many heterogeneous agent distributions and this is certainly true of the model of Krusell and Smith (1998). Moreover, Huggett's result ensures that this is a lower bound. As the steady state value for capital increases, the interest rate falls. From (20), the rise in wealth due to higher aggregate capital is more than offset by the increase in discounted labor earnings. Therefore, one can easily calculate the corresponding $\omega^{\star}$ associated with an x% increase in the steady state.

Finally, we view this thought experiment as a *first step* in understanding how a model aggregates. It is not meant to replace rigorous analysis (we are not advocating for the "lazy" economist) but provides a good assessment of aggregation at an *extremely low cost*. For example, a researcher might be interested in understanding how perturbations to certain parameters affect aggregation. Steady state analysis is a good place to start.

4.2 Assessing the Finite-Moment Auctioneer By far the most popular auctioneer used to numerically solve heterogeneous agent models is what we call the Finite-Moment Auctioneer. Initially proposed in the early working papers of Krusell and Smith (1998), Castaneda, Díaz-Giménez, and Ríos-Rull (1998) and den Haan (1993), this auctioneer forecasts aggregate capital in future periods by using a law of motion which depends only on aggregate capital in the current period and the current aggregate state. This formulation generalizes to the Higher-Order Moments Auctioneer which forecasts capital using further distributional statistics, such as the variance, in addition to the mean. The numerical implementation of the Finite-Moment Auctioneer is well known and therefore only briefly described here. Our algorithm follows the standard stochastic-simulation approach of Krusell and Smith (1998) (KS, henceforth).[15]

Standard Algorithm

1. Guess an initial savings function, aggregate law of motion, and cross-sectional distribution of households. Generate a long sequence of total factor productivity shocks once and for all.
2. Solve the household's problem by Euler equation iteration or value function iteration, beginning from the initial savings function and using the aggregate law of motion to forecast one-period-forward prices. Iterate until the savings functions converge up to some tolerance.
3. Use the savings function from Step 2 to simulate the cross-sectional distribution for the sequence of TFP shocks generated in Step 1 via the procedure of Young (2010).
4. Use the simulated data generated in Step 3 to update the aggregate law of motion by ordinary least squares regression (in the case of a law which is linear in coefficients).
5. Repeat steps 2-4 until the aggregate law of motion converges within some tolerance.
6. Test for equilibrium. For example, one can compute the R^2 fit of the regression in Step 4, or use the procedure advocated by den Haan (2010).

In this section, we focus on interpreting the auctioneer output of Steps 4–6. Remarkably—and to the best of our knowledge—the literature does not contain an interpretation of the regression output along the lines proposed here. We show that approximate aggregation is *not* the key insight, nor the driving force of this algorithm, as claimed by Krusell and Smith (2006). We then use this interpretation to offer an improvement of Step 6 in the following section.

[15] The algorithm is more aptly described as a modified version of the stochastic-simulation algorithm due to Maliar, Maliar, and Valli (2010), which combines that paper's household solving method with the simulation procedure of Young (2010). For all results reported below, the algorithm was implemented in the programming language Julia. Interpolation in Step 2 uses cubic splines via the Julia package Dierckx, which acts as a wrapper for the FORTRAN package of the same name.

Interpreting Regression Output Initially, we set $\delta = 1$ as complete depreciation of capital permits cleaner exposition, and assume the standard calibration $\{\sigma = 1; \delta = 1; \alpha = 0.36; \beta = 0.99; T \to \infty\}$ with the stochastic processes for technology and the labor market following KS. The converged equilibrium law of motion for the two aggregate states from 1,000 simulated draws is computed to be

$$\log(K') = -0.0090 + 0.3605 \log(K), \quad R^2 = 1, \qquad s = \textit{good} \tag{22}$$

$$\log(K') = 0.0522 + 0.3603 \log(K), \quad R^2 = 1, \qquad s = \textit{bad} \tag{23}$$

The R^2 values contain eight 9's after the decimal, so we rounded up.

Defining Ω_t as the share of aggregate output invested, $K_t = \Omega_t Y_t$, it is well known that the representative agent equilibrium with log utility and complete depreciation of capital can be calculated in closed form. Under this scenario, the share of aggregate investment is time invariant and given by $\Omega = \alpha\beta$. Thus we have $K_t = \alpha\beta Y_t = \alpha\beta K_{t-1}^{\alpha} L_t^{1-\alpha}$ and taking logs gives

$$\log(K_t) = \log(\alpha\beta) + \log(z_t) + \alpha \log(K_{t-1}) \tag{24}$$

where we have assumed the representative agent inelastically supplies one unit of aggregate labor in each period. Simulating data from (24) would yield a nearly deterministic law of motion for aggregate capital. Running regressions (22)–(23) on this simulated data would deliver an R^2 of approximately one, slope coefficient of α, and intercept of $\log(\alpha\beta) + \log(z_j)$. With the exception of the intercept term, (24) is a good approximation of (22)–(23).[16]

The intuition of this result dates back at least to the optimal portfolio literature of Merton (1969) and Samuelson (1969). With log utility and i.i.d. returns, the optimal portfolio, expressed as a percentage of wealth, is independent of the level of wealth. Complete depreciation of capital breaks the time dependence of returns and the technology shock merely alters the level of wealth in the economy. The optimal share of investment (Ω) is constant and the coefficient of determination (R^2) is identically one.

This intuition carries over to heterogeneous agent models, which can be seen by rewriting the

[16] The R^2 values are not exactly one and there are slight differences in the coefficients across states because the idiosyncratic shock is correlated with the aggregate shock in the KS environment, and because aggregate labor is not identical across the two regimes. The intercept term is not well approximated because there is no labor uncertainty in the standard representative agent model. A better approximation of (22)–(23) that takes labor uncertainty into account is given by the linear-nonlinear decomposition of Section 3. Defining the average share of (un)employed households born at time t as (s_u^t) s_e^t, then the linear component of aggregate capital is $K_t = s_u^t k_{u,t} + s_e^t k_{e,t}$,

$$K_t(1+\beta) = \beta s_e^t W_t^{(g)} L_t^{(g)} - (s_u^t \pi_{ggue} + s_e^t \pi_{ggee}) \left(\frac{W_{t+1}^{(g)} L_{t+1}^{(g)}}{R_{t+1}^{(g)}} \right) - (s_u^t \pi_{gbue} + s_e^t \pi_{gbee}) \left(\frac{W_{t+1}^{(b)} L_{t+1}^{(b)}}{R_{t+1}^{(b)}} \right)$$

where the idiosyncratic transition probabilities are conditional on aggregate states following Krusell and Smith (1998): π_{ggue} represents the probability of the aggregate state remaining "good" and transitioning from unemployed (u) to employed (e).

household's budget constraint (2) in terms of shares,

$$\begin{aligned} c_t &= R_t k_{t-1} + W_t \ell_t - k_t = R_t K_{t-1} s_{t-1} + W_t L_t e_t - \Omega_t Y_t s_t \\ &= (\alpha s_{t-1} + (1-\alpha) e_t - \Omega_t s_t) Y_t \end{aligned} \tag{25}$$

where s_t is the household's share of aggregate investment $s_t K_t = k_t$ and e_t is the household's share of the aggregate wage bill. Substituting the constraint into the log utility function reveals an optimal savings / consumption allocation that is independent of the *level* of wealth. That is, taking logs of (25) separates the choice variable, s_t, from aggregate output, Y_t. That the optimal share of investment for each household is independent of the level of wealth implies that the aggregate investment share (Ω) will also be independent of the level of wealth, and therefore time invariant. Households care only about their share of the pie and not the overall size of the pie, which is changing with aggregate shocks. This result is important in our analysis of aggregation because it holds independent of aggregate wealth and therefore holds *independent of aggregation*. Wealth can be sufficiently low such that aggregation fails and the aggregate investment share (Ω) will remain constant, implying R^2 values will remain close to one. This can be seen from our results in (22)–(23). The R^2 values for the complete depreciation case are *higher* than the standard calibration (results reported in footnote 14) despite the fact that approximate aggregation is no longer holding as strongly. We provide additional evidence for this result in the following section.

When $\delta \neq 1$, the optimal amount of aggregate capital is a function of total resources, including undepreciated capital, $K_t = \Omega_t(Y_t + (1-\delta)K_{t-1}) = \Omega_t(z_t K_{t-1}^{\alpha} L_t^{1-\alpha} + (1-\delta)K_{t-1})$. The share of resources invested, Ω_t, now varies with the level of wealth as the return to capital is not time independent. Thus, the law of motion for capital will not be linear in logs. However, taking advantage of the approximation $\log(1+x) \approx x$ for small x, we have

$$\begin{aligned} \ln(K_t) &= \ln(\Omega_t) + \ln(K_{t-1}) + \ln(z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} + (1-\delta)) \\ &\approx \ln(\Omega_t) + \ln(K_{t-1}) + z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} - \delta \end{aligned} \tag{26}$$

where $z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} - \delta \approx 0.07$ for $K \approx 40, z = L = 1$, which are the steady state values for these variables. From (26), the regression slope coefficient will be slightly less than one and the intercept term will be close to zero, which is consistent with standard results.[17] That the R^2 remains close to one—which is also a standard result—implies that there is not much time variation in the share of output invested (Ω). While the share of output invested is no longer constant as in the complete depreciation case, conditioning on the aggregate state (Ω_j) implies a near constant value set by the auctioneer. The upshot here is that R^2 values are not necessarily tied to aggregation. Aggregation

[17] The original KS regressions give

$$\begin{aligned} &\log(K') = 0.095 + 0.962\log(K), \quad R^2 = 0.999998, \qquad s = \mathit{good} \\ &\log(K') = 0.085 + 0.965\log(K), \quad R^2 = 0.999998, \qquad s = \mathit{bad} \end{aligned}$$

can fail and the R^2 values can be close to one; conversely, a representative agent model can generate "low" R^2 values.

In order to carefully assess the algorithm, we require a setup in which aggregation is not the base case. The importance of T highlighted in Section 3.3 suggests a particular model structure that preserves the theory of Section 3 while providing the best chance of breaking aggregation.[18] Specifically, time is discrete and infinite. In each period, a measure $1/T$ of T-period lived households is born, so that at any given time there is a unit measure of living individuals. Households behave according to the theory described in Section 3. Equilibrium for this economy will now involve a distribution $\Lambda_s(k, \ell, t)$ at time s of households across capital k, employment ℓ, and age $t = 1, ..., T$. For each time s, we initialize the new cohort by fixing $\Lambda_s(0, 0, 1) = u_s/T$ and $\Lambda_s(0, 1, 1) = (1 - u_s)/T$, where u_s is the average unemployment rate corresponding to the aggregate state at time s. Households born into unemployment are given a small unemployment benefit (μ), which is paid by taxing the wage of current workers. This overlapping-generations, life-cycle framework has several advantages. First, as $T \to \infty$, it replicates well-known environments (Aiyagari (1994), Krusell and Smith (1998)). Second, for $T = 2$, we are able to study an exact equilibrium; that is, one that is not subject to the curse of dimensionality. This facilitates a precise evaluation of common numerical approaches. Third, it is a natural environment for breaking aggregation as households representing different generations have different marginal propensities to save, as documented above. Finally, an infinite horizon economy permits the study of simulation methods and steady state analysis. Thus, we can use the KS algorithm described above to solve the model.

Assessment of Step 6: Testing for Equilibrium The literature continues to rely on regression statistics such as the coefficient of determination (R^2) as a test of the equilibrium (Step 6), despite den Haan's (2010) compelling counter-examples that show arbitrarily high R^2 values may not correspond to accurate aggregate laws of motion. We now show that low R^2 values do not always imply inaccurate laws of motion.

Example 1: Let $T = 2$ and simulate an exact sequence of equilibrium capital values $\{K_t\}_{t=1}^N$ using the guess-and-verify algorithm in Appendix B. Define the auctioneer's law of motion as a function of investment, $I_t \equiv K_t - (1 - \delta)K_{t-1}$, as opposed to log capital; that is, estimate $I_t = a_{0,j} + I_{t-1}a_{1,j}$ for $j = \{g, b\}$ with the simulated data. The corresponding R^2 values are 0.0786024 (good) and 0.110295 (bad). Any researcher using the current standards for Step 6 would clearly reject this particular law of motion. However, the accuracy of this "investment auctioneer" is not nearly as poor as the R^2 suggests. Since investment is defined as δ-differenced capital, the auctioneer is estimating a flow variable (investment) as opposed to a stock (capital). The fit of the regression will be significantly lower but it does not imply that *economically* meaningful estimates are impossible to achieve. Using the estimated law of motion for investment to forecast next period's aggregate capital, the maximum error is only 1.6% and the average error is 0.6%. This auctioneer would provide extremely accurate values of tomorrow's aggregate capital (and hence tomorrow's interest

[18] The model and numerical approach to solving it are described in Appendix B.

rate) to households, despite having an R^2 value close to zero.

Example 2: Consider another example that uses our knowledge of an exact equilibrium ($T = 2$): From the previous section, we established that the share of output invested is nearly time invariant when $T = 2$, implying an R^2 very close to one. We follow the standard methodology for solving and evaluating the equilibrium. Specifically, we define idiosyncratic grid points distributed on an interval around the deterministic steady state according to the polynomial rule $k_j = (j/100)^7 K^{\max}$, for $j = 1, ..., 100$, while four aggregate grid points are distributed linearly on an interval around the deterministic steady state. Convergence criteria for the log linear aggregate law of motion was set to $1e^{-6}$. The R^2 values of the converged law of motion are roughly 0.99997 for both states. The mean/max Den-Haan forecast errors are 0.003%/0.01%, respectively.[19] These are clearly acceptable values. However, mean capital is roughly 37% below the actual equilibrium average, and the simulated equilibrium paths never cross. Experienced computational economists will perform several additional robustness checks to find that increasing the number of aggregate grid points substantially changes the equilibrium. For example as the number of aggregate grid points is increased to 100, the R^2 approaches 0.999999997, the mean/max Den-Haan error falls to 4.2e-5/1.3e-4, and the actual equilibrium is accurately approximated. Those with insufficient experience may view the initial R^2 / Den-Haan statistics as acceptable values, and poorly approximate the true equilibrium.

Example 3: This example follows from the results in Krueger and Kubler (2006). Aggregation interacts with the regression statistics in important ways. If we plot the corresponding R^2 values against T using the standard log-linear law of motion (22)–(23), we would get a u-shaped figure with the *minimum* occurring at $T = 9$. While the R^2 values are lowest where aggregation is least likely to hold, these values are still quite high (0.9977, 0.9978 for $T = 9$). Moreover, adding moments to the regression, which is the standard operating procedure when R^2 values fall, does not alter the equilibrium substantially. For values of $7 \leq T \leq 15$, the maximum difference in the mean capital stock between the "mean only" regression and "mean plus variance" regression is less than 3%. Krueger and Kubler (2006) advocate for a superior projection / Smolyak algorithm that better fits the evolving distribution of wealth.

The issue that our examples highlight is that the current set of "convergence" statistics do not give cut-off points instructing users when to search for alternative computational strategies. Consequently, the statistics currently used to test for an equilibrium can lead to erroneous conclusions. Both type I and type II errors are possible: rejecting a perfectly reasonable guess for the aggregate law of motion (Example 1) and accepting a law of motion that is quite far from the true equilibrium (Examples 2 and 3). This is true for both the R^2 values and the more-recent Den-Haan statistics.

We propose taking advantage of the dichotomous nature of the problem to assess the accuracy of the auctioneer. Conditional on a guess for the auctioneer's values, the household's problem is

[19] In the context of the KS model, den Haan (2010) defines forecast errors as $u_{t+1} \equiv |\hat{K}_{t+1} - K_{t+1}|$, where $\hat{K}_{t+1}$ is simulated from the estimated law of motion (e.g., (22)–(23)), and K_{t+1} is obtained by drawing a new sequence of exogenous shocks and simulating the model.

extremely well posed: Proposition 1 proves existence and uniqueness conditions; each household's problem can be solved in isolation, implying an embarrassingly-parallel computing strategy can be deployed; and finally, the solution to the households' problem provides an arbitrarily precise evaluation of the auctioneer through the market clearing mechanism. Any guess of the auctioneer's values must be consistent with the market-clearing behavior of the individual households. This suggests an iterative procedure for solving the auctioneer's problem.

4.3 Improving the Finite-Moment Auctioneer Our suggested numerical procedure follows Steps 1–5 of the standard algorithm but replaces Step 6 with the following:

Updated Step 6: Using the simulated values for the (converged) aggregate capital stock $\{K_t^0\}_{t=1}^N$ and the converged law of motion for aggregate capital

$$\log(K') = \alpha^0_{0,j} + \alpha^0_{1,j}\log(K), \qquad j = \{\text{good}, \text{bad}\} \tag{27}$$

produced by Steps 1–5, generate a sequence of one-step-ahead forecasts for interest rates and wages $\{R_t^0, W_t^0\}_{t=2}^N$ from the firm's first-order conditions. Using the interest rate and wage sequence $\{R_t^0, W_t^0\}_{t=2}^N$, repeat steps 1 to 5 to obtain a new aggregate series $\{K_t^1\}_{t=1}^N$. If this series is within tolerance of $\{K_t^0\}_{t=1}^N$, stop. Otherwise, replace $\{K_t^0\}_{t=1}^N$ with $\{K_t^1\}_{t=1}^N$ and repeat.

This evaluation step replaces model-dependent statistics—like the R^2—with an iterative procedure for the auctioneer. Therefore, convergence criteria can be applied to the auctioneer's problem in much the same way that it is applied to the household's problem. It views the output coming from Steps 1–5 as a "potential solution," to be used as the initial sequence in the iteration. In the notation of the household's problem defined in Sections 2.5–2.4, the initial sequence and aggregate law of motion are a guess for the auctioneer predictions, $\mathcal{F}$, which is the interest rate and wage sequence $\{R_t^0, W_t^0\}_{t=2}^N$. As discussed in those sections, the auctioneer's guess must satisfy optimality and market clearing conditions. Thus, the iterative procedure disciplines the auctioneer's guess along these lines.

From a numerical standpoint, the Updated Step 6 replaces the aggregate grid with the simulated values for aggregate capital and prices. The entire empirical distribution of capital is therefore evaluated. The iteration essentially operationalizes the D-H forecast errors, which are defined as $u_{t+1} \equiv |\hat{K}_{t+1} - K_{t+1}|$, where $\hat{K}_{t+1}$ is simulated from the estimated law of motion (e.g., (22)–(23)), and K_{t+1} is obtained by solving the household's problem. Step 6 states that if these errors are too large, an update of the auctioneer's sequence is warranted. The downside of the Updated Step 6 is that the model must be solved again. However, we do not view this as a major obstacle given the computational ease with which the household's problem can be solved[20], and potential efficiency gains may be made by employing a relatively coarse histogram of the simulated empirical distribution.

[20]The speed of the Updated Step 6 depends upon the number of simulated values N and optimality of the initial aggregate grid. The values reported in Table 1 for $T = 4$ and $N = 1,000$ took 63 seconds using Matlab's Parallel Computing Toolbox on a Windows 10 machine with an Intel Xeon 3.50 GHz processor, 4 physical cores and 32 GB of RAM. Note that the model was solved in Matlab, not Julia, for this exercise.

	$T=2$			$T=3$			$T=4$		
	Stand.	Add Var.	Auc. Iter.	Stand.	Add Var.	Auc. Iter.	Stand.	Add Var.	Auc. Iter.
R^2 (good)	0.9999	0.9999	0.9999	0.9647	0.9790	0.9985	0.9816	0.9912	0.9999
R^2 (bad)	0.9999	0.9999	0.9999	0.9790	0.9824	0.9974	0.9895	0.9934	0.9995
D-H (max)	0.0237	0.0131	$1.3e^{-7}$	0.0117	0.0117	$3.2e^{-7}$	0.1596	0.1325	$4.0e^{-7}$
D-H (mean)	0.0097	0.0056	$1.2e^{-8}$	0.0020	0.0021	$4.5e^{-7}$	0.0252	0.0235	$5.6e^{-7}$
% Avg Dev	37.59%	26.93%	$9.2e^{-5}$	30.43%	26.23%	$8.9e^{-8}$	15.21%	15.13%	$9.6e^{-9}$

Table 1: Values of the coefficient of determination (R^2), Den-Haan statistics (D-H), and percentage deviation from equilibrium (% Mean Dev.) are reported for the Standard Algorithm (Stand.), Standard Algorithm plus Second Moment (Add Var.), and the Auctioneer Iteration (Auc. Iter.) for $T = 2, 3, 4$.

Table 1 demonstrates the usefulness of adding an iterative check on the auctioneer's problem for $T = 2, 3, 4$. The table reports the converged solution and diagnostics for the Standard Algorithm (Stand.), the Standard Algorithm with an additional second moment added to the aggregate law of motion (Add Var.), and the Standard Algorithm with Step 6 replaced with the Updated Step 6 (Auc. Iter.), with an auctioneer iteration tolerance of $1e^{-8}$. As in Example 2, the algorithms were implemented with sparse aggregate grids (six aggregate grid points) which generated equilibria that were far from the true bounded-rational equilibrium yet with reasonable values for the R^2 and D-H statistics. The last row of Table 1 reports the percentage deviation of average aggregate capital from the true value, which was obtained through a well-placed, dense aggregate grid (64 grid points).

Without an iterative procedure for the auctioneer's problem, it is impossible to determine when the R^2 statistic is sufficiently close to one and when the D-H statistics are sufficiently small. For each value of T and especially for $T = 2$, the R^2 values are quite high and the D-H statistics are low but the equilibrium remains far from the true value for the standard algorithm. The current operating procedure is to add moments to the aggregate law of motion when the regression fit is sufficiently poor. However, adding the second-moment does not dramatically improve the performance of the algorithm. The Updated Step 6 (Auctioneer Iteration) replaces the sparse aggregate grid with the simulated values of aggregate capital generated by the standard algorithm. In all cases ($T = 2, 3, 4$), the updated algorithm finds the more accurate solution.

It is perhaps more telling when the Auctioneer Iteration does *not* converge. For values of T greater than six, the updated algorithm does not converge without a substantial reduction in auctioneer tolerance. This is a clear indication that the guess for the auctioneer's law of motion needs modification. In these examples, the R^2 and D-H statistics do fall but not substantially so (recall that the trough in R^2 does not occur until $T = 9$). Again, without an iterative procedure with set tolerance, it is impossible to know when an R^2 value or D-H statistic is "too low." The Updated Step 6 provides much clearer guidance.

5 Concluding Thoughts

We examined uninsurable employment risk with aggregate shocks by introducing a Walrasian auctioneer, who reports to households all possible state-contingent future prices. Households take these as given when forming expectations and making optimal consumption choices, and the auctioneer adjusts her forecasts until markets clear. By taking advantage of this natural dichotomy between the households and the auctioneer, we studied each problem in isolation. On the household side, we separate an explicit expression for the linear permanent income component of savings from a well-behaved nonlinear adjustment arising from precautionary behavior and incomplete markets. Equipped with this decomposition, we then study how economies aggregate in the presence of various auctioneer types that are popular in the literature. We provided an economic interpretation of the regression coefficients and explained the lack of time variation in the auctioneer of Krusell and Smith (1998). We also introduced a new numerical method which uses the empirical distribution of auctioneer forecasts to substantially improve solution accuracy in cases where the standard coefficient of determination and other well-known statistics prove to be misleading.

References

Acharya, S., and K. Dogra (2020): "Understanding HANK: Insights from a PRANK," *Econometrica*, 88(3), 1113–1158.

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2021): "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," Forthcoming, *Review of Economic Studies*.

Aiyagari, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109(3), 659–684.

Aiyagari, S. R., and M. Gertler (1991): "Asset Returns with Transactions Costs and Uninsured Individual Risk," *Journal of Monetary Economics*, 27, 311–331.

Auclert, A. (2019): "Monetary Policy and the Redistribution Channel," *American Economic Review*, 109(6), 2333–2367.

Bewley, T. (1982): "Thoughts on Tests of the Intertemporal Asset Pricing Model," Working paper, Northwestern University.

Bilbiie, F. O. (2008): "Limited Asset Markets Participation, Monetary Policy and (inverted) Aggregate Demand Logic," *Journal of Economic Theory*, 140, 162–196.

——— (2018): "Heterogeneity, Determinacy, and New Keynesian Puzzles," Working Paper.

——— (2021): "Monetary Policy and Heterogeneity: An Analytical Framework," Working Paper.

Boppart, T., P. Krusell, and K. Mitman (2018): "Exploiting MIT Shocks in Heterogeneous-Agent Economies: The Impulse response as a Numerical Derivative," *Journal of Economic Dynamics and Control*, 89, 68–92.

Carroll, C. (2014): "Theoretical Foundations of Buffer Stock Saving," NBER Working Paper 10867.

Carroll, C. D. (1992): "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, pp. 61–135.

——— (1997): "Buffer-Stock Saving and the Life Cycle / Permanent Income Hypothesis," *Quarterly Journal of Economics*, 112, 1–55.

Carroll, C. D., and M. Kimball (1996): "On the Concavity of the Consumption Function," *Econometrica*, 64, 981–992.

Castaneda, A., J. Díaz-Giménez, and J.-V. Ríos-Rull (1998): "Exploring the Income Distribution Business Cycle Dynamics," *Journal of Monetary Economics*, 42(1), 93–130.

Chatterjee, S. (1994): "Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model," *Journal of Public Economics*, 54(1), 97–119.

Constantinides, G. M. (1982): "Intertemporal Asset Pricing with Heterogeneous Consumers and Without Demand Aggregation," *Journal of Business*, 55, 253–267.

Constantinides, G. M., and D. Duffie (1996): "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104, 219–240.

Cooley, T., R. Marimon, and V. Quadrini (2004): "Aggregate Consequences of Limited Contract Enforceability," *Journal of Political Economy*, 112(4), 817–847.

Cooley, T., and V. Quadrini (2006): "Monetary Policy and the Financial Decisions of Firms," *Economic Theory*, 27(1), 243–270.

Deaton, A. (1991): "Saving and Liquidity Constraints," *Econometrica*, 59, 1221–1248.

Debortoli, D., and J. Galí (2018): "Monetary Policy with Heterogeneous Agents: Insights from TANK models," Working paper, CREI.

den Haan, W. J. (1993): "Solving Heterogeneous Agent Models: An Application to Asset Pricing With Incomplete Markets," unpublished manuscript, Department of Economics, UCSD.

——— (2010): "Assessing the Accuracy of the Aggregate Law of Motion in Models with Heterogeneous Agents," *Journal of Economic Dynamics and Control*, 34(1), 79–99.

den Haan, W. J., K. L. Judd, and M. Juillard (2010): "Computational Suite of Models with Heterogeneous agents: Incomplete Markets and Aggregate Uncertainty," *Journal of Economic Dynamics and Control*, 34(1), 1–3.

den Haan, W. J., and P. Rendahl (2010): "Solving the Incomplete Markets Model with Aggregate Uncertainty using Explicit Aggregation," *Journal of Economics Dynamics and Control*, 34, 69–78.

Favilukis, J., S. C. Ludvigson, and S. V. Nieuwerburgh (2017): "The Macroeconomic Effects of Housing Wealth, Housing Finance, and Limited Risk Sharing in General Equilibrium," *Journal of Political Economy*, 125(1).

Friedman, M. (1957): "The Permanent Income Hypothesis," in *A Theory of the Consumption Function*, pp. 20–37. Princeton University Press.

Gorman, W. M. (1953): "Community Preference Fields," *Econometrica*, 21, 63–80.

——— (1961): "On a Class of Preference Fields," *Metroeconomica*, 13, 53–56.

Gornemann, N., K. Kuester, and M. Nakajima (2016): "Doves for the Rich, Hawks for the Poor? Distributional Consequences of Monetary Policy," International Finance Discussion Papers 1167, Board of Governors of the Federal Reserve System.

Gourinchas, P.-O., and J. A. Parker (2002): "Consumption over the Life Cycle," *Econometrica*, 70, 47–89.

Guvenen, F. (2011): "Macroeconomics with Heterogeneity: A Practical Guide," NBER Working Paper No. 17622.

Heathcote, J., K. Storesletten, and G. L. Violante (2009): "Quantitative Macroeconomics with Heterogeneous Households," NBER Working Paper No. 14768.

Heaton, J., and D. J. Lucas (1992): "The Effects of Incomplete Insurance Markets and Trading Costs in a Consumption-Based Asset Pricing Model," *Journal of Economic Dynamics and Control*, 16, 601–620.

Huggett, M. (1993): "The Risk-Free Rate in Heterogeneous-Agent, Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17, 953–970.

——— (1997): "The One-Section Growth Model with Idiosyncratic Shocks: Steady States and Dynamics," *Journal of Monetary Economics*, 39, 385–403.

Kaplan, G., B. Moll, and G. L. Violante (2018): "Monetary Policy According to HANK," *American Economic Review*, 108, 697–743.

Khan, A., and J. K. Thomas (2013): "Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity," *Journal of Political Economy*, 121(6), 1055–1107.

Krueger, D., and F. Kubler (2006): "Computing Equilibrium in OLG Models with Stochastic Production," *Journal of Economic Dynamics and Control*, 28, 1411–1436.

Krueger, D., K. Mitman, and F. Perri (2016): "Macroeconomics and Household Heterogeneity," vol. 2 of *Handbook of Macroeconomics*, pp. 843 – 921. Elsevier.

Krusell, P., and A. A. Smith, Jr (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.

——— (2006): "Quantitative Macroeconomic Models with Heterogeneous Agents," in *Advances in Economics and Econometrics: Theory and Applications*, ed. by T. P. R. Blundell, W. Newey. Cambridge University Press.

Levine, D. K., and W. R. Zame (2002): "Does Market Incompleteness Matter?," *Econometrica*, 70, 1805–1839.

Ljungqvist, L., and T. J. Sargent (2004): *Recursive Macroeconomic Theory*. MIT Press, Cambridge, MA, 2nd edn.

Lucas, D. (1994): "Asset Pricing with Undiversifiable Income Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle," *Journal of Monetary Economics*, 34, 325–341.

Lucas, J. R. E. (1990): "Liquidity and Interest Rates," *Journal of Economic Theory*, 50(April), 237–264.

Maliar, L., S. Maliar, and F. Valli (2010): "Solving the Incomplete Markets Model with Aggregate Uncertainty using the Krusell–Smith Algorithm," *Journal of Economic Dynamics and Control*, 34(1), 42–49.

McKay, A., E. Nakamura, and J. Steinsson (2016): "The Power of Forward Guidance Revisited," *American Economic Review*, 106, 3133–58.

Merton, R. C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics*, 51, 247–257.

Prescott, E. C., and R. Mehra (1980): "Recursive Competitive Equilibrium: The Case of Homogeneous Households," *Econometrica*, 48(6), 1365–1379.

Ravn, M. O., and V. Sterk (2020): "Macroeconomic Fluctuations with HANK & SAM: An Analytical Approach," forthcoming, *Journal of the European Economic Association*.

Reiter, M. (2010): "Solving the Incomplete Markets Model with Aggregate Uncertainty by Backward Induction," *Journal of Economic Dynamics and Control*, 34(1), 28 – 35.

Samuelson, P. A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *The Review of Economics and Statistics*, 51(51), 239–246.

Storesletten, K., C. I. Telmer, and A. Yaron (2007): "Asset Pricing with Idiosyncratic Risk and Overlapping Generations," *Review of Economic Dynamics*, 10(4), 519–548.

Straub, L. (2018): "Consumption, Savings, and the Distribution of Permanent Income," Working paper, MIT.

Takahashi, S. (2014): "Heterogeneity and Aggregation: Implications for Labor-Market Fluctuations: Comment," *The American Economic Review*, 104, 1446–1460.

Telmer, C. I. (1993): "Asset-Pricing Puzzles and Incomplete Markets," *Journal of Finance*, 48, 1803–32.

Werning, I. (2015): "Incomplete Markets and Aggregate Demand," *National Bureau of Economic Research*, Working Paper 21448.

Winberry, T. (2018): "A Method for Solving and Estimating Heterogeneous Agent Macro Models," *Quantitative Economics*, 9, 1123–1151.

Yaari, M. (1976): "A Law of Large Numbers in the Theory of Consumer's Choice under Uncertainty," *Journal of Economic Theory*, 12, 202–217.

Young, E. R. (2010): "Solving the Incomplete Markets Model with Aggregate Uncertainty using the Krusell-Smith Algorithm and Non-Stochastic Simulations," *Journal of Economic Dynamics and Control*, 34(1), 36–41.

Zeldes, S. P. (1989): "Consumption and Liquidity Constraints: An Empirical Investigation," *Journal of Political Economy*, 97, 305–346.

Zhang, L. (2005): "The Value Premium," *Journal of Finance*, 60(1), 67–103.

6 Appendix A (Not for Publication)

In this appendix next we provide rigorous proofs of our theoretical results leading up to the main theorems. We begin with Proposition 1, which establishes existence and uniqueness for the households' dynamic programming problems. This proof is a straightforward exercise in finite horizon dynamic programming. For convenience, we recall the statement here.

Proposition 1: Household Existence and Uniqueness. There is a unique solution to the household's dynamic programming problem (6). The associated savings functions $k^{(t)}$ are increasing (strictly for $t < T$) with respect to x_t and satisfies

$$\lim_{x_t \to \underline{k}_t} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \underline{k}_t,$$

$$\lim_{x_t \to \infty} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \infty, \quad t < T$$

The corresponding value functions are strictly increasing and strictly concave with respect to ω and satisfy

$$\lim_{x_t \to \underline{k}_t} V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = -\infty$$

Proof. The proof is an induction, beginning with the terminal period T. Since $V^{(T+1)} \equiv 0$, it is immediate that the unique solution to the terminal problem is $c^{(T)}(x_T, \mathcal{L}_t, \mathcal{F}_t) = x_T$ and $k^{(T)} \equiv 0$, with corresponding value function $V^{(T)} = u(x_T)$. The savings function is trivially increasing, while the value function is strictly increasing and strictly concave by our selection of preferences. Moreover, in this case we have $\underline{k}_T = 0$, and so that the stated limits at the borrowing constraint hold, trivially in the case of $k^{(T)}$ and due to the asymptote of the period utility function in the case of $V^{(T)}$.

Having established the base case, we now suppose that, given $t < T$, we have a unique solution $(c^{(t+1)}, k^{(t+1)}, V^{(t+1)})$ satisfying the stated properties. As discussed in the main text, the natural borrowing limit implies that the first order conditions are necessary and sufficient for a solution to the household problem. Writing this condition in period t, we have

$$\frac{1}{(x_t - k_t)^\sigma} = \beta \mathbb{E}_t (1 - \delta + R_{t+1}) \frac{\partial V^{(t+1)}(x_{t+1}, \mathcal{L}_{t+1}, \mathcal{F}_{t+1})}{\partial x_{t+1}}$$

For a fixed value of x_t, the left side of this equation is strictly increasing in $k_t \in (\underline{k}_t, x_t)$, and increases without bound as $k_t \to x_t^-$. On the right side, the limit at $\underline{k}_t$ is ∞ due to the asymptote in the value function at the borrowing constraint. Moreover, since x_{t+1} is strictly increasing in k_t, strict concavity of the value function in resources implies that the right side of the equation is strictly decreasing. It follows from these observations that there is a unique value $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$ such that the equation balances.

To see that the savings function is increasing, we differentiate implicitly the first order condition to get

$$\frac{-\sigma}{(x_t - k_t)^{\sigma+1}}\left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right) = \beta\mathbb{E}_t(1-\delta+R_{t+1})^2\left(\frac{\partial^2 V}{\partial x_{t+1}^2}\right)\frac{\partial k^{(t)}}{\partial x_t}$$

Solving for the derivative of the savings function, we have

$$\left(\frac{\sigma}{(x_t - k_t)^{\sigma+1}} - \beta\mathbb{E}_t(1-\delta+R_{t+1})^2\left(\frac{\partial^2 V}{\partial x_{t+1}^2}\right)\right)\frac{\partial k^{(t)}}{\partial x_t} = \frac{\sigma}{(x_t - k_t)^{\sigma+1}}$$

Once again applying concavity of the value function, we see that all terms here are positive, which establishes (strict) monotonicity.

The limit of the savings function at resources amounting to the natural borrowing limit follows immediately from the squeeze theorem, since $\underline{k}_t \leq k^{(t)} \leq x_t$. We can translate this inequality to read

$$0 \leq x_t - k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \leq x_t - \underline{k}_t$$

which show in turn that $c^{(t)} \to 0$ as resources approach the borrowing limit. Then, writing

$$V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = u\left(x_t - k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)\right) + \beta\mathbb{E}_t V^{(t+1)}(x_{t+1}, \mathcal{L}_{t+1}, \mathcal{F}_{t+1})$$

the asymptote for the period t value function follows from inspection of the first term. The limit of the savings function at ∞ is evident from the Euler equations (7)-(8) along with the definition of x_{t+1}. In particular, if household resources in period t are increased without bound, $x_t \to \infty$, while household savings (and hence period $t+1$ resources) remain bounded, the left side of the period t Euler equation would vanish while the right side remained strictly positive, a contradiction.

Monotonicity and convexity of the value function follow from the envelope conditions. Explicitly, differentiating the above expression

$$\begin{aligned}
\frac{\partial V^{(t)}}{\partial x_t} &= \frac{1}{(x_t - k^{(t)})^{\sigma}}\left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right) + \beta\mathbb{E}_t\frac{\partial V^{(t+1)}}{\partial x_{t+1}}(1-\delta+R_{t+1})\frac{\partial k^{(t)}}{\partial x_t} \\
\frac{\partial^2 V^{(t)}}{\partial x_t^2} &= \frac{-\sigma}{(x_t - k^{(t)})^{\sigma+1}}\left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right)^2 + \beta\mathbb{E}_t\frac{\partial^2 V^{(t+1)}}{\partial x_{t+1}^2}(1-\delta+R_{t+1})^2\left(\frac{\partial k^{(t)}}{\partial x_t}\right)^2 \\
&\quad - \frac{1}{(x_t - k^{(t)})^{\sigma}}\frac{\partial^2 k^{(t)}}{\partial x_t^2} + \beta\mathbb{E}_t\frac{\partial V^{(t+1)}}{\partial x_{t+1}}(1-\delta+R_{t+1})\frac{\partial^2 k^{(t)}}{\partial x_t^2}
\end{aligned}$$

Using first order conditions to simplify these, we therefore get (respectively)

$$\frac{\partial V^{(t)}}{\partial x_t} = \frac{1}{(x_t - k^{(t)})^\sigma} > 0$$

$$\frac{\partial^2 V^{(t)}}{\partial x_t^2} = \frac{-\sigma}{(x_t - k^{(t)})^{\sigma+1}} \left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right)^2 + \beta \mathbb{E}_t \frac{\partial^2 V^{(t+1)}}{\partial x_{t+1}^2} (1 - \delta + R_{t+1})^2 \left(\frac{\partial k^{(t)}}{\partial x_t}\right)^2 < 0$$

as desired. This closes the induction. □

Next, we derive the savings function in the deterministic environment with a natural borrowing limit. We will do so in the case of general risk aversion parameters σ. First we recall the expression in the form of a proposition.

Proposition 2: Savings in a Deterministic Production Economy. Let

$$Q_{T-1} = \left[\beta(1 - \delta + R_T)^{1-\sigma}\right]^{1/\sigma}$$

$$Q_t = \left[\beta(1 - \delta + R_{t+1})^{1-\sigma}\right]^{1/\sigma} (1 + Q_{t+1}), \quad t = 1, ..., T - 2$$

In the production economy without uncertainty, the household savings function is given by

$$k_t(x_t) = \frac{Q_t}{1 + Q_t} x_t - \frac{1}{1 + Q_t} \left(\sum_{s=t+1}^{T} \frac{W_s \ell}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right)$$

Proof. Taking σth roots in the terminal Euler equation, we obtain

$$\begin{aligned} x_{T-1} - k_{T-1} &= \frac{x_T}{(\beta(1 - \delta + R_T))^{1/\sigma}} \\ &= \frac{(1 - \delta + R_T) k_{T-1} + W_T \ell}{(\beta(1 - \delta + R_T))^{1/\sigma}} \\ &= \frac{k_{T-1} + \frac{W_T \ell}{1 - \delta + R_T}}{(\beta(1 - \delta + R_T)^{1-\sigma})^{1/\sigma}} \end{aligned}$$

Solving for k_{T-1} gives

$$k_{T-1} = \frac{\left(\beta(1 - \delta + R_T)^{1-\sigma}\right)^{1/\sigma}}{1 + (\beta(1 - \delta + R_T)^{1-\sigma})^{1/\sigma}} x_{T-1} - \frac{1}{1 + (\beta(1 - \delta + R_T)^{1-\sigma})^{1/\sigma}} \frac{W_T \ell}{1 - \delta + R_T}$$

In terms of the definition of Q_{T-1} and the savings function, this says

$$k^{(T-1)}(x_t) = \frac{Q_{T-1}}{1 + Q_{T-1}} x_{T-1} - \frac{1}{1 + Q_{T-1}} \frac{W_T \ell}{1 - \delta + R_T}$$

We can now proceed by induction, using the formula for the period $t+1$ savings function to simplify the period t Euler equation. We can write this equation as

$$x_t - k_t = \frac{k_t + \frac{W_{t+1}\ell}{1-\delta+R_{t+1}} - \frac{k_{t+1}}{1-\delta+R_{t+1}}}{\left(\beta(1-\delta+R_{t+1})^{1-\sigma}\right)^{1/\sigma}}$$

This will be satisfied taking k_t and k_{t+1} given by the period t and $t+1$ savings functions, respectively. Using the inductive hypothesis for the latter, with $x_{t+1} = (1-\delta+R_{t+1})k_t + W_{t+1}\ell$, we can rewrite the Euler equation as

$$\begin{aligned} x_t - k_t &= \frac{\frac{1}{1+Q_{t+1}}k_t + \frac{1}{1+Q_{t+1}}\frac{W_{t+1}\ell}{1-\delta+R_{t+1}} + \frac{1}{1-\delta+R_{t+1}}\frac{1}{1+Q_{t+1}}\left(\sum_{s=t+2}^{T}\frac{W_s\ell}{\prod_{r=t+2}^{s}(1-\delta+R_r)}\right)}{\left(\beta(1-\delta+R_{t+1})^{1-\sigma}\right)^{1/\sigma}} \\ &= \frac{\frac{1}{1+Q_{t+1}}k_t + \frac{1}{1+Q_{t+1}}\sum_{s=t+1}^{T}\frac{W_s\ell}{\prod_{r=t+1}^{s}(1-\delta+R_r)}}{\left(\beta(1-\delta+R_{t+1})^{1-\sigma}\right)^{1/\sigma}} \end{aligned}$$

Solving for k_t, we obtain the desired expression. □

Next, we fill in the details of Theorem 1. This version of the theorem admits direct calculations which will be replaced by more circumspect arguments as we allow for more generality. We begin from the rearranged Euler equation (33).

Theorem 1: Nonlinear error, Log Utility. The savings function $k^{(1)}(x_1)$ can be written in the form

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\mathbb{E}\,\ell_2 + \epsilon^{(1)}(x_1)\right)$$

where the nonlinear error term is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1, \qquad \lim_{x_1\to\infty}\epsilon^{(1)}(x_1) = 0$$

Proof. Having established (33) via the calculations in Section 3.2, we may express the right side as

$$\begin{aligned} &k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}}) \\ &\quad + \frac{\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right)}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]} - \left(k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\right) \end{aligned}$$

Writing the last two terms as a single fraction gives

$$\frac{\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right) - \left(k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\right)\left(k_1 + \frac{W_2}{1-\delta+R_2}\left(p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right)\right)}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]}$$

which after some cancellation in the numerator yields

$$\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{\ell_{\text{low}}\ell_{\text{high}} - (p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\,(p\ell_{\text{high}} + (1-p)\ell_{\text{low}})}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]}$$

so that the right side of (33) is now

$$\begin{aligned} k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}}) \\ + \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{\ell_{\text{low}}\ell_{\text{high}} - (p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\,(p\ell_{\text{high}} + (1-p)\ell_{\text{low}})}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]} \end{aligned}$$

The numerator in the trailing expression can be written as

$$\begin{aligned} \ell_{\text{low}}\ell_{\text{high}} - (p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\,(p\ell_{\text{high}} + (1-p)\ell_{\text{low}}) \\ &= (1-p^2-(1-p)^2)\ell_{\text{low}}\ell_{\text{high}} - p(1-p)(\ell_{\text{high}}^2 + \ell_{\text{low}}^2) \\ &= (2p-2p^2)\ell_{\text{low}}\ell_{\text{high}} - p(1-p)(\ell_{\text{high}}^2 + \ell_{\text{low}}^2) \\ &= -p(1-p)\,(\ell_{\text{high}} - \ell_{\text{low}})^2 \end{aligned}$$

We can no write out (33) in its entirety as

$$\begin{aligned} &\beta(x_1 - k_1) \\ &= k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}}) - \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\,(\ell_{\text{high}} - \ell_{\text{low}})^2}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]} \end{aligned}$$

Collecting the terms linear in k_1, this rearranges to give (34). As alluded to in the main text, then, we apply Proposition 1 to conclude that there is a well-defined savings function $k^{(1)}$ which satisfies this equation and the conditions of that proposition. Letting

$$\epsilon^{(1)} = \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\,(\ell_{\text{high}} - \ell_{\text{low}})^2}{k^{(1)} + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]} \tag{28}$$

we have

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \epsilon^{(1)}(x_1)\right) \tag{29}$$

We must now establish properties of the nonlinear error term. The limit of this error term at the borrowing constraint follows from the limit of the savings function at this constraint. Specifically, taking the limit in the above expression we have

$$\underline{k}_1 = \frac{1}{1+\beta}\left(\beta\underline{k}_1 - \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1)\right)$$

which we solve for

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1$$

To calculate the limit as resources increase without bound, we apply our knowledge that savings increase without bound in this limit along with the definition (28) to get

$$\begin{aligned}\lim_{x_1\to\infty}\epsilon^{(1)}(x_1) &= \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\lim_{x_1\to\infty}k^{(1)} + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]} \\ &= 0\end{aligned}$$

To see monotonicity, observe that

$$\frac{\partial\epsilon^{(1)}(x_1)}{\partial x_1} = -\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)} + \frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}]\right)^2}\frac{\partial k^{(1)}}{\partial x_1} < 0 \tag{30}$$

where the inequality follows from the fact that the savings function is increasing.

To see convexity, first observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(1)}}{\partial x_1^2} = \frac{\partial^2\epsilon^{(1)}}{\partial x_1^2} \tag{31}$$

Then, taking a second derivative in (30) we have

$$\begin{aligned}\frac{\partial^2\epsilon^{(1)}(x_1)}{\partial x_1^2} &= 2\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)} + \frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}]\right)^3}\frac{\partial k^{(1)}}{\partial x_1} \\ &\quad - \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)} + \frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}]\right)^2}\frac{\partial^2 k^{(1)}}{\partial x_1^2}\end{aligned}$$

Using (31) and solving, we get

$$\frac{\partial^2\epsilon^{(1)}(x_1)}{\partial x_1^2} = 2\frac{\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\right)^3}}{1+\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\right)^2}}\frac{\partial k^{(1)}}{\partial x_1}$$

Every expression on the right hand side is positive, which gives convexity of the nonlinear error term. This completes the proof. $\square$

In order to prove our main theorems, we need the following lemma. While this lemma amounts to a basic calculus exercise, it will be a vital ingredient in general versions of our theorems.

Lemma 1. *Let $A, B \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\lim_{x\to\infty} (f(x) - Ax) = B$$

Then

$$\lim_{x\to\infty} (f(x) - (Ax + B)) = 0.$$

Proof. We have

$$\lim_{x\to\infty} (f(x) - (Ax + B)) = \lim_{x\to\infty} (f(x) - Ax) - B = B - B = 0$$

which proves it. $\square$

On a practical note, observe that

$$\lim_{x\to\infty} \left(\frac{f(x)}{x} - A\right) = \lim_{x\to\infty} \left(\frac{f(x) - Ax}{x}\right) = \lim_{x\to\infty} \left(\frac{B}{x}\right) = 0$$

Rearranging gives us a simple way to calculate the slope A:

$$A = \lim_{x\to\infty} \frac{f(x)}{x}$$

We are now ready to complete the proof of Theorem 1, in which we have omitted aggregate uncertainty and restricted ourselves to two periods, while allowing for general period utility.

Theorem 2: Nonlinear Error, CRRA Utility. The savings function $k^{(1)}(x_1)$ with log replaced by a general CRRA utility function can be written in the form

$$k^{(1)}(x_1) = \frac{1}{1+Q_1}\left(Q_1 x_1 - \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 + \epsilon^{(1)}(x_1)\right),$$
$$\text{with} \quad Q_1 = \left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}$$

where the nonlinear error term is strictly decreasing, convex, and satisfies

$$\lim_{x_1\to\underline{k}_1} \epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1, \qquad \lim_{x_1\to\infty} \epsilon^{(1)}(x_1) = 0$$

Proof. To simplify our expressions, we will adopt the notation

$$w_{\text{low}} = \frac{W_2}{1-\delta+R_2}\ell_{\text{low}}, \quad w_{\text{high}} = \frac{W_2}{1-\delta+R_2}\ell_{\text{high}}$$

throughout the proof.

We begin from the rearranged Euler equation, which we restate here for convenience:

$$\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}(x_1-k_1) = \frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p\,(k_1+w_{\text{high}})^{\sigma}+(1-p)\,(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}$$

We will apply Lemma 1 to the right hand side of this equation. To this end, we calculate the limit

$$\lim_{k_1\to\infty}\left(\frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p\,(k_1+w_{\text{high}})^{\sigma}+(1-p)\,(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}-k_1\right)$$

Expressing the argument as a single fraction, rearranged, we wish to calculate

$$\lim_{k_1\to\infty}\left(\frac{k_1^2\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)+k_1(w_{\text{low}}+w_{\text{high}})+w_{\text{low}}w_{\text{high}}}{k_1\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}}\right)$$

We split the argument into three separate fractions, one for each term in the numerator, and compute the limit of each. After some thought, the third of these limits is zero, while the second is $w_{\text{low}}+w_{\text{high}}$. The first limit we rewrite as

$$\lim_{k_1\to\infty}\left(\frac{k_1\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)}{\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}}\right)$$

The denominator here has limit 1, so the above limit will equal

$$\lim_{k_1\to\infty}\left(k_1\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)\right)$$

provided this limit exists. We rewrite the argument as

$$\lim_{k_1\to\infty}\left(\frac{\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)}{1/k_1}\right)$$

and observe that the numerator and denominator both have limit equal to zero as $k_1\to\infty$. We may therefore use l'Hospital's rule to conclude that this limit is equal to that of

$$\frac{-\frac{1}{\sigma}\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{\frac{1}{\sigma}-1}}{(-1/k_1^2)} \times\left[\sigma p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma-1}\left(-\frac{w_{\text{high}}}{k_1^2}\right)+\sigma(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma-1}\left(-\frac{w_{\text{low}}}{k_1^2}\right)\right]$$

This expression simplifies to give

$$-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{\frac{1}{\sigma}-1} \times\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma-1}w_{\text{high}}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma-1}w_{\text{low}}\right]$$

from which we can read off the limit as

$$-pw_{\text{high}}-(1-p)w_{\text{low}}$$

Combining all three limits, we now get

$$\begin{aligned}\lim_{k_1\to\infty}\left(\frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p(k_1+w_{\text{high}})^{\sigma}+(1-p)(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}-k_1\right)&=-pw_{\text{high}}-(1-p)w_{\text{low}}+w_{\text{low}}+w_{\text{high}}\\&=\mathbb{E}_1w_2\end{aligned}$$

Applying Lemma 1, we conclude that

$$\lim_{k_1\to\infty}\left(\frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p(k_1+w_{\text{high}})^{\sigma}+(1-p)(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}-k_1-\mathbb{E}_1w_2\right)=0$$

Denoting the argument of the limit by $-\delta(k_1)$, we may therefore write the Euler equation in this case as

$$\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}(x_1-k_1)=k_1+\mathbb{E}_1w_2-\delta(k_1)$$

with $\lim_{k_1\to\infty}\delta(k_1)=0$. Rearranging, this becomes

$$k_1=\frac{1}{1+Q_1}\left(Q_1x_1-\mathbb{E}_1w_2+\delta(k_1)\right)$$

with Q_1 as given in the theorem statement.

Once again invoking the existence proposition as in the proof of the previous theorem and letting $\epsilon^{(1)}(x_1) = \delta\left(k^{(1)}(x_1)\right)$, we obtain the desired decomposition

$$k^{(1)}(x_1) = \frac{1}{1+Q_1}\left(Q_1 x_1 - \mathbb{E}_1 w_2 + \epsilon^{(1)}(x_1)\right)$$

Taking limits at the borrowing constraint,

$$\underline{k}_1 = \frac{1}{1+Q_1}\left(Q_1 \underline{k}_1 - \mathbb{E}_1 w_2 + \lim_{x_1 \to \underline{k}_1} \epsilon^{(1)}(x_1)\right)$$

which we solve for

$$\lim_{x_1 \to \underline{k}_1} \epsilon^{(1)}(x_1) = \mathbb{E}_1 w_2 + \underline{k}_1$$

We also have

$$\lim_{x_1 \to \infty} \epsilon^{(1)}(x_1) = \lim_{x_1 \to \infty} \delta\left(k^{(1)}(x_1)\right) = 0$$

since $k^{(1)}(x_1) \to \infty$ as $x_1 \to \infty$ (by the existence and uniqueness proposition) and $\delta \to 0$ as its argument increases to ∞.

To establish that $\epsilon^{(1)}$ is decreasing in resources, we observe that

$$\frac{\partial \epsilon^{(1)}}{\partial x_1} = \frac{\partial \delta}{\partial k_1}\frac{\partial k^{(1)}}{\partial x_1}$$

has the same sign as $\partial \delta / \partial k_1$. Next we observe that the definition of $\delta(k_1)$ satisfies

$$\left(\frac{p}{(k_1 + w_{\text{low}})^\sigma} + \frac{1-p}{(k_1 + w_{\text{high}})^\sigma}\right)^{-1/\sigma} = k_1 + \mathbb{E}_1 w_2 - \delta(k_1) \tag{32}$$

It therefore suffices to show that the derivative of the left side of the above equality with respect to k_1 is larger than unity. Calculating this derivative, it suffices to show that

$$\left(\frac{p}{(k_1 + w_{\text{low}})^\sigma} + \frac{1-p}{(k_1 + w_{\text{high}})^\sigma}\right)^{-\frac{1}{\sigma}-1}\left(\frac{p}{(k_1 + w_{\text{low}})^{\sigma+1}} + \frac{1-p}{(k_1 + w_{\text{high}})^{\sigma+1}}\right) > 1$$

Rearranging, this is equivalent to the bound

$$\left(\frac{p}{(k_1 + w_{\text{low}})^\sigma} + \frac{1-p}{(k_1 + w_{\text{high}})^\sigma}\right)^{\frac{1}{\sigma}} < \left(\frac{p}{(k_1 + w_{\text{low}})^{\sigma+1}} + \frac{1-p}{(k_1 + w_{\text{high}})^{\sigma+1}}\right)^{\frac{1}{\sigma+1}}$$

That this inequality is true is a consequence of the fact that L^σ norms are increasing in σ (which in turn follows from Jensen's inequality).

The proof of convexity is similar. First, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(1)}}{\partial x_1^2} = \frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2}$$

so that

$$\begin{aligned}
\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2} &= \frac{\partial^2 \delta}{\partial k_1^2}\left(\frac{\partial k^{(1)}}{\partial x_1}\right)^2 + \frac{\partial \delta}{\partial k_1}\frac{\partial^2 k^{(1)}}{\partial x_1^2} \\
&= \frac{\partial^2 \delta}{\partial k_1^2}\left(\frac{\partial k^{(1)}}{\partial x_1}\right)^2 + \frac{\partial \delta}{\partial k_1}\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2}
\end{aligned}$$

and consequently

$$\left(1 - \frac{\partial \delta}{\partial k_1}\right)\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2} = \frac{\partial^2 \delta}{\partial k_1^2}\left(\frac{\partial k^{(1)}}{\partial x_1}\right)^2$$

Since we have seen above that δ is decreasing in k_1 it suffices to show that δ is convex.

Letting

$$\begin{aligned}
h(k_1) &:= \frac{p}{(k_1 + w_{\text{low}})^\sigma} + \frac{1-p}{(k_1 + w_{\text{high}})^\sigma} \\
g(k_1) &:= h(k_1)^{-1/\sigma}
\end{aligned}$$

we note that

$$\frac{\partial^2 \delta}{\partial k_1^2} = -\frac{\partial^2 g}{\partial k_1^2}$$

so that it suffices to show that $\partial^2 g/\partial k_1^2 < 0$. We have

$$\begin{aligned}
\frac{\partial g}{\partial k_1} &= -\frac{1}{\sigma}h(k_1)^{-\frac{1}{\sigma}-1}\frac{\partial h}{\partial k_1} \\
\frac{\partial^2 g}{\partial k_1^2} &= -\frac{1}{\sigma}\left(-\frac{1}{\sigma}-1\right)h(k_1)^{-\frac{1}{\sigma}-2}\left(\frac{\partial h}{\partial k_1}\right)^2 - \frac{1}{\sigma}h(k_1)^{-\frac{1}{\sigma}-1}\frac{\partial^2 h}{\partial k_1^2} \\
&= \frac{1}{\sigma}h(k_1)^{-\frac{1}{\sigma}-2}\left(\left(\frac{1}{\sigma}+1\right)\left(\frac{\partial h}{\partial k_1}\right)^2 - h(k_1)\frac{\partial^2 h}{\partial k_1^2}\right)
\end{aligned}$$

It therefore suffices that the expression in brackets in the last line is negative. To do so, we first observe that

$$\frac{\partial h}{\partial k_1} = \frac{-\sigma p}{(k_1 + w_{\text{low}})^{\sigma+1}} + \frac{-\sigma(1-p)}{(k_1 + w_{\text{high}})^{\sigma+1}}$$
$$\frac{\partial^2 h}{\partial k_1^2} = \frac{\sigma(\sigma+1)p}{(k_1 + w_{\text{low}})^{\sigma+2}} + \frac{\sigma(\sigma+1)(1-p)}{(k_1 + w_{\text{high}})^{\sigma+2}}$$

so it suffices to show that

$$\left(\frac{1}{\sigma}+1\right)\left[\frac{-\sigma p}{(k_1 + w_{\text{low}})^{\sigma+1}} + \frac{-\sigma(1-p)}{(k_1 + w_{\text{high}})^{\sigma+1}}\right]^2 - \left[\frac{p}{(k_1 + w_{\text{low}})^{\sigma}} + \frac{1-p}{(k_1 + w_{\text{high}})^{\sigma}}\right]\left[\frac{\sigma(\sigma+1)p}{(k_1 + w_{\text{low}})^{\sigma+2}} + \frac{\sigma(\sigma+1)(1-p)}{(k_1 + w_{\text{high}})^{\sigma+2}}\right] < 0$$

Clearing factors of σ and rearranging, this is equivalent to showing

$$\left[\frac{p}{(k_1 + w_{\text{low}})^{\sigma+1}} + \frac{(1-p)}{(k_1 + w_{\text{high}})^{\sigma+1}}\right]^2 < \left[\frac{p}{(k_1 + w_{\text{low}})^{\sigma}} + \frac{1-p}{(k_1 + w_{\text{high}})^{\sigma}}\right]\left[\frac{p}{(k_1 + w_{\text{low}})^{\sigma+2}} + \frac{(1-p)}{(k_1 + w_{\text{high}})^{\sigma+2}}\right]$$

That this inequality holds is a consequence of the Cauchy-Schwartz inequality. This completes the proof of the theorem. □

Proof of Proposition: Recall that the Euler equation (8) is necessary for optimality with the natural borrowing constraint. Assuming log utility, we write the expected value out explicitly to obtain

$$\frac{1}{x_1 - k_1} = \beta\left(\frac{p(1-\delta+R_2)}{(1-\delta+R_2)k_1 + W_2\ell_{\text{low}}} + \frac{(1-p)(1-\delta+R_2)}{(1-\delta+R_2)k_1 + W_2\ell_{\text{high}}}\right)$$

Taking reciprocals and rearranging the aggregate quantities under the expected value rewrites this as

$$\beta(x_1 - k_1) = \left(\frac{p}{k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}} + \frac{(1-p)}{k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}}\right)^{-1}$$

The left hand side is an expression which is linear in x_1 and k_1, while the right hand side is necessarily nonlinear in k_1 due to the idiosyncratic uncertainty. Nonetheless, we can add the fractions under the brackets to obtain

$$\beta(x_1 - k_1) = \frac{\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right)}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]} \tag{33}$$

This form clarifies the structure of the right hand side: it consists of a rational function in k_1 which is formed from the ratio of a quadratic polynomial to a linear one. The asymptotic behavior of such a function is such that it approaches a linear asymptote as $k_1 \to \infty$. With some additional algebra, we can extract this asymptote, rewriting the above as

$$\begin{aligned} k_1 = & \left(\frac{\beta}{1+\beta}\right) x_1 - \frac{1}{1+\beta}\left(\frac{W_2}{1-\delta+R_2}\right)[p\ell_{\text{low}} + (1-p)\ell_{\text{high}}] \\ & + \frac{1}{1+\beta}\left(\frac{W_2}{1-\delta+R_2}\right)^2 \left(\frac{p(1-p)(\ell_{\text{high}}-\ell_{\text{low}})^2}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]}\right) \end{aligned} \tag{34}$$

To facilitate interpretation of this expression, let $n_2 := \frac{W_2}{1-\delta+R_2}(\ell - \ell_{\text{low}})$ denote the household's discounted excess earnings so that

$$n_2 = \begin{cases} 0 & \text{with probability } p \\ \frac{W_2}{1-\delta+R_2}(\ell_{\text{high}} - \ell_{\text{low}}) & \text{with probability } 1-p \end{cases}$$

Then a straightforward calculation gives

$$\begin{aligned} \operatorname{Var}(n_2) &= p(1-p)\left[\frac{W_2}{1-\delta+R_2}(\ell_{\text{high}} - \ell_{\text{low}})\right]^2 \\ \frac{W_2}{1-\delta+R_2}\ell_{\text{low}} + \frac{\operatorname{Var}(n_2)}{\mathbb{E}(n_2)} &= \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right] \end{aligned}$$

so that our rearranged Euler equation gives

$$k_1 = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\mathbb{E}\,\ell_2 + \left(\frac{\operatorname{Var}(n_2)}{k_1 + \frac{W_2}{1-\delta+R_2}\ell_{\text{low}} + \frac{\operatorname{Var}(n_2)}{\mathbb{E}(n_2)}}\right)\right)$$

We now give details of the proofs of our main theorems. The proof of the logarithmic case illustrates the majority of the main features, while eliminating the somewhat tedious tracking of recursive effective discount factors which appear in the general case. For this reason, we present this case in detail before giving the argument in full generality.

We will find it convenient to introduce the notation $o(f(x))$ to denote any function $g(x)$ such that $\lim_{x\to\infty} g(x)/f(x) = 0$. In particular, $o(1)$ denotes any function $g(x)$ such that $\lim_{x\to\infty} g(x) = 0$.

Recall the statement of the theorem in the log case.

Theorem 3: Main Theorem, $\sigma = 1$. The savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ which solve the dynamic programming problems (6) with $\sigma = 1$ can be written in the form

$$\begin{aligned} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = & \frac{\beta + ... + \beta^{T-t}}{1+\beta+...+\beta^{T-t}} x_t - \frac{1}{1+\beta+...+\beta^{T-t}} \mathbb{E}_t\left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) \\ & + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \end{aligned}$$

where the nonlinear error term $\epsilon^{(t)}$ is identically zero without uncertainty, and is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1 \to \underline{k}_t} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{1}{1+\beta+...+\beta^{T-t}} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)} \right) + \underline{k}_t \right]$$

and

$$\lim_{x_1 \to \infty} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = 0$$

with uncertainty.

Proof. The proof is by induction, beginning with the period $T-1$ savings function. The base case essentially follows the argument given for the two period model detailed in Appendix I, albeit with additional bookkeeping due to the inclusion of arbitrarily many shock outcomes and aggregate uncertainty. Although it is lengthy, we provide the argument, to illustrate the algebra without the need to worry about additional endogeneity due to future savings.

Base Case (T-1): We begin by writing out the expected value of the terminal Euler equation (8). To facilitate this, for a given state $(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})$ of resources and predictive probabilities at time $T-1$ we let

$$\{(\ell_j, R_j, W_j) : \ j = 1, ..., J\}$$

denote the possible time T endowment and price outcomes which are assigned positive predictive probability. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j / (1-\delta+R_j)$.

With this notation in hand the Euler equation can be written out as

$$\begin{aligned} \frac{1}{x_{T-1} - k_{T-1}} &= \beta \left(\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j} \right) \\ &= \beta \left(\frac{\sum_{j=1}^{J} p_j \prod_{i \neq j}(k_{T-1} + w_i)}{\prod_j (k_{T-1} + w_j)} \right) \end{aligned}$$

Taking the reciprocal, this gives

$$\beta(x_{T-1} - k_{T-1}) = \frac{\prod_j (k_{T-1} + w_j)}{\sum_{j=1}^{J} p_j \prod_{i \neq j}(k_{T-1} + w_i)} \tag{35}$$

The right hand side is a rational function whose numerator has degree J and whose denominator has degree $J-1$. We therefore expect this to approach some linear asymptote as $k_{T-1} \to \infty$. To extract the asymptote, we write out the numerator and denominator, getting

$$\beta(x_{T-1} - k_{T-1}) = \frac{k_{T-1}^J + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})}$$

We can now apply the lemma on the right hand side in a transparent way. Specifically, we have

$$\begin{aligned}
&\lim_{k_{T-1}\to\infty} \left(\frac{k_{T-1}^J + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})} - k_{T-1} \right) \\
&\qquad = \lim_{k_{T-1}\to\infty} \left(\frac{k_{T-1}^J + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1}) - k_{T-1}^J + k_{T-1}^{J-1} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})} \right) \\
&\qquad = \lim_{k_{T-1}\to\infty} \left(\frac{k_{T-1}^{J-1} \sum_j p_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})} \right) \\
&\qquad = \lim_{k_{T-1}\to\infty} \left(\frac{\sum_j p_j w_j + o(1)}{1 + o(1)} \right) \\
&\qquad = \sum_j p_j w_j \\
&\qquad = \mathbb{E}_{T-1} w_T
\end{aligned}$$

so that the lemma says that

$$\lim_{k_{T-1}\to\infty} \left(\frac{\prod_j (k_{T-1} + w_j)}{\sum_{j=1}^J p_j \prod_{i \neq j} (k_{T-1} + w_i)} - k_{T-1} - \mathbb{E}_{T-1} w_T \right) = 0$$

Letting $-\delta(k_{T-1})$ denote the argument of the above limit, we combine with (35) to get

$$\beta(x_{T-1} - k_{T-1}) = k_{T-1} + \mathbb{E}_{T-1} w_T - \delta(k_{T-1})$$

By the existence and uniqueness proposition, there is a unique period $T-1$ savings function $k^{(T-1)}$ which solves the household problem in this period. Hence we may define

$$\epsilon^{(1)}(x_1, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) := \delta\left(k^{(1)}(x_1, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\right)$$

and rearrange the above equation into the desired form

$$k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \frac{1}{1+\beta} \left(\beta x_{T-1} - \mathbb{E}_{T-1} w_T + \epsilon^{(1)}(x_1, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) \right)$$

We can calculate the behavior of the nonlinear error at the domain endpoints identically to the two period CRRA case above. Precisely, taking limits at the borrowing constraint,

$$\underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \frac{1}{1+\beta}\Bigg(\beta \underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1}) - \mathbb{E}_{T-1} w_T + \lim_{x_{T-1} \to \underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1})} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\Bigg)$$

which we solve for

$$\lim_{x_{T-1} \to \underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1})} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \mathbb{E}_{T-1} w_T + \underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1})$$

We also have

$$\lim_{x_{T-1} \to \infty} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \lim_{x_{T-1} \to \infty} \delta\left(k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\right) = 0$$

since $k^{(T-1)}(x_{T-1}) \to \infty$ as $x_{T-1} \to \infty$ (by the existence and uniqueness proposition) and $\delta \to 0$ as its argument increases to ∞.

To establish that $\epsilon^{(T-1)}$ is decreasing in resources, we once again observe that

$$\frac{\partial \epsilon^{(T-1)}}{\partial x_{T-1}} = \frac{\partial \delta}{\partial k_{T-1}} \frac{\partial k^{(T-1)}}{\partial x_{T-1}}$$

has the same sign as $\partial \delta / \partial k_{T-1}$. Next we observe that the definition of $\delta(k_{T-1})$ satisfies

$$\left(\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}\right)^{-1} = k_{T-1} + \mathbb{E}_{T-1} w_T - \delta(k_{T-1})$$

It therefore suffices to show that the derivative of the left side of the above equality with respect to k_{T-1} is larger than unity. Calculating this derivative, it suffices to show that

$$\left(\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}\right)^{-2} \left(\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2}\right) > 1$$

Rearranging, this is equivalent to the bound

$$\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j} < \left(\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2}\right)^{1/2}$$

That this inequality is true is a consequence of the fact that the L^1 norm on the left is dominated by the L^2 norm on the right (which, again, follows from Jensen's inequality).

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(T-1)}}{\partial x_{T-1}^2} = \frac{\partial^2 \epsilon^{(T-1)}}{\partial x_{T-1}^2}$$

so that, exactly as in the two period CRRA case, it suffices to show that δ is convex. Letting

$$\begin{aligned} h(k_{T-1}) &:= \sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j} \\ g(k_{T-1}) &:= 1/h(k_{T-1}) \end{aligned}$$

and repeating the derivative calculations in that case, we see that it suffices to show that

$$2\left(\frac{\partial h}{\partial k_{T-1}}\right)^2 - h(k_{T-1})\frac{\partial^2 h}{\partial k_{T-1}^2} < 0$$

To do so, we calculate

$$\begin{aligned} \frac{\partial h}{\partial k_{T-1}} &= -\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2} \\ \frac{\partial^2 h}{\partial k_{T-1}^2} &= 2\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^3} \end{aligned}$$

so it suffices to show that

$$\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2}\right]^2 - \left[\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}\right]\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^3}\right] < 0$$

which is equivalent to

$$\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2}\right]^2 < \left[\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}\right]\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^3}\right]$$

This inequality is once again true by the Cauchy-Schwartz inequality, and this establishes our base case for the induction.

Inductive Step. Suppose now that the theorem is proved for the savings function in period $t+1$, where $1 < t \leq T-1$. For a given state $(x_t, \mathcal{L}_t, \mathcal{F}_t)$ of resources and predictive probabilities at time t, we suppose that there are J time $t+1$ states which are assigned positive predictive probability, and we let

$$\{(\ell_j, R_j, W_j) : \ j = 1, ..., J\}$$

denote the possible time $t+1$ endowment and price outcomes. Note that both J and the set in the above line may change depending on which period t we are focused on - we suppress these dependencies for simplicity of notation. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j / (1-\delta+R_j)$.

The inductive step will, naturally, involve what we already know about the time $t+1$ savings function and its nonlinear error term. For simplicity, we will suppress the dependence of these functions on predictive distributions in the argument.

Having established these notational conventions the time t Euler equation can be written as

$$\frac{1}{x_t - k_t} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{k_t + w_j - \frac{1}{1-\delta+R_j} k_{t+1}} \right)$$

For optimality, time $t+1$ savings must be given by the time $t+1$ savings function, and by the inductive hypothesis we have

$$\begin{aligned}
&\frac{1}{1-\delta+R_j} k^{(t+1)}\Big((1-\delta+R_j)k_t + W_j\ell_j\Big) \\
&\qquad = \frac{\beta + \ldots + \beta^{T-t-1}}{1+\beta+\ldots+\beta^{T-t-1}}(k_t + w_j) - \frac{1}{1+\beta+\ldots+\beta^{T-t-1}} \mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)} \right) \\
&\qquad + \epsilon^{(t+1)}\Big((1-\delta+R_j)k_t + W_j\ell_j\Big)
\end{aligned}$$

so that the denominators on the right side of the Euler equation take the form

$$\begin{aligned}
&\frac{1}{1+\beta+\ldots+\beta^{T-t-1}} k_t + \frac{1}{1+\beta+\ldots+\beta^{T-t-1}} \left(w_j + \mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)} \right)\right) \\
&\qquad - \epsilon^{(t+1)}\Big((1-\delta+R_j)k_t + W_j\ell_j\Big)
\end{aligned}$$

In order to simplify the algebraic steps to follow, we let

$$\begin{aligned}
A_j \equiv A &:= \frac{1}{1+\beta+\ldots+\beta^{T-t-1}} \\
B_j &:= \frac{1}{1+\beta+\ldots+\beta^{T-t-1}} \left(w_j + \mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)} \right)\right) \\
C_j(k_t) &:= B_j - \epsilon^{(t+1)}\Big((1-\delta+R_j)k_t + W_j\ell_j\Big)
\end{aligned}$$

In this notation, the Euler equation becomes

$$\frac{1}{x_t - k_t} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)} \right)$$

We now proceed much as in the terminal case, by writing this Euler equation as

$$\beta(x_t - k_t) = \frac{\prod_{j=1}^{J}(Ak_t + C_j(k_t))}{\sum_{j=1}^{J} p_j \prod_{i\neq j}(Ak_t + C_i(k_t))}$$

In order to apply Lemma 1, we now compute

$$\lim_{k_t\to\infty}\left(\frac{\prod_{j=1}^{J}(Ak_t + C_j(k_t))}{\sum_{j=1}^{J} p_j \prod_{i\neq j}(Ak_t + C_i(k_t))} - Ak_t\right)$$

We may rewrite the argument here as a single fraction. To do so, we first note that

$$\lim_{k_t\to\infty}\frac{C_j(k_t)}{k_t} = \lim_{k_t\to\infty}\left(\frac{B_j}{k_t} - \frac{\epsilon^{(t+1)}\Big((1-\delta+R_j)k_t + W_j\ell_j\Big)}{k_t}\right) = 0$$

so that $C_j(k_t) = o(k_t)$. After some cancellation, then this single fraction can be written as

$$\begin{aligned}
&\frac{A^{J-1}k_t^{J-1}\sum_j C_j(k_t) + o(k^{J-1}) - A^{J-1}k_t^{J-1}\sum_{j=1}^{J} p_j \sum_{i\neq j} C_i(k_t) + o(k^{J-1})}{\sum_{j=1}^{J} p_j \prod_{i\neq j}(Ak_t + C_i(k_t))}\\
&\qquad = \frac{A^{J-1}k_t^{J-1}\sum_j p_j C_j(k_t) + o(k^{J-1})}{A^{J-1}k_t^{J-1} + o(k^{J-1})}\\
&\qquad = \frac{\sum_j p_j C_j(k_t) + o(1)}{1 + o(1)}\\
&\qquad \longrightarrow \sum_j p_j C_j(k_t) = \mathbb{E}_t C_{t+1}(k_t) \text{ as } k_t \to \infty
\end{aligned}$$

Noting that

$$\begin{aligned}
\mathbb{E}_t C_{t+1}(k_t) &= \mathbb{E}_t \frac{1}{1+\beta+\ldots+\beta^{T-t-1}}\left(w_{t+1} + \mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T}\frac{W_s\ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)}\right)\right)\\
&\quad - \mathbb{E}_t\epsilon^{(t+1)}\Big((1-\delta+R_{t+1})k_t + W_{t+1}\ell_{t+1}\Big)\\
&= \frac{1}{1+\beta+\ldots+\beta^{T-t-1}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) + o(1)
\end{aligned}$$

Lemma 1 tells us that

$$\lim_{k_t\to\infty}\left(\beta\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t + C_j(k_t)}\right) - Ak_t - \frac{1}{1+\beta+\ldots+\beta^{T-t-1}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right)\right) = 0$$

Letting $-\delta(k_t)$ denote the argument of this limit, it follows that we can rewrite the time t Euler equation as

$$\beta(x_t - k_t) = Ak_t + \frac{1}{1+\beta+...+\beta^{T-t-1}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) - \delta(k_t)$$

with $\lim_{k_t\to\infty}\delta(k_t) = 0$. Recalling the definition of A and rearranging, this gives

$$k_t = \frac{\beta+...+\beta^{T-t}}{1+\beta+...+\beta^{T-t}}x_t - \frac{1}{1+\beta+...+\beta^{T-t}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) + \delta(k_t)$$

As in previous arguments, we define

$$\epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) := \delta\left(k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)\right)$$

which gives us the desired form of the savings function.

The limits of the nonlinear error follow as usual: the limit at the borrowing constraint from rearranging the identity

$$\begin{aligned}\underline{k}_t(\mathcal{L}_t, \mathcal{F}_t) = {} & \frac{\beta+...+\beta^{T-t}}{1+\beta+...+\beta^{T-t}}\underline{k}_t(\mathcal{L}_t, \mathcal{F}_t) - \frac{1}{1+\beta+...+\beta^{T-t}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) \\ & + \lim_{x_t\to\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t)}\epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)\end{aligned}$$

and the limit at ∞ from the definition of $\epsilon^{(t)}$ above, the behavior of $k^{(t)}$ at ∞, and the definition of δ.

By a similar argument to previous cases, monotonicity of the error term will follow from showing that the derivative of

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t + C_j(k_t)}\right)^{-1}$$

with respect to k_t is larger than A.

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t + C_j(k_t)}\right)^{-2}\left(\sum_{j=1}^{J}\frac{p_j(A + \frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2}\right) > A$$

Noting that

$$\frac{\partial C_j}{\partial k_t} = -(1-\delta+R_j)\frac{\partial\epsilon^{(t+1)}}{\partial x_{t+1}} > 0$$

we have

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t+C_j(k_t)}\right)^{-2}\left(\sum_{j=1}^{J}\frac{p_j(A+\frac{\partial C_j}{\partial k_t})}{(Ak_t+C_j(k_t))^2}\right)>\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t+C_j(k_t)}\right)^{-2}\left(\sum_{j=1}^{J}\frac{p_jA}{(Ak_t+C_j(k_t))^2}\right)$$

so it is sufficient to show that

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t+C_j(k_t)}\right)^{-2}\left(\sum_{j=1}^{J}\frac{p_jA}{(Ak_t+C_j(k_t))^2}\right)\geq A$$

which is implied by showing

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t+C_j(k_t)}\right)^{-2}\left(\sum_{j=1}^{J}\frac{p_j}{(Ak_t+C_j(k_t))^2}\right)\geq 1$$

Rearranging, we get

$$\sum_{j=1}^{J}\frac{p_j}{Ak_t+C_j(k_t)}\leq\left(\sum_{j=1}^{J}\frac{p_j}{(Ak_t+C_j(k_t))^2}\right)^{1/2}$$

which is true by the norm argument given in the base case.

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(t)}}{\partial x_t^2}=\frac{\partial^2 \epsilon^{(t)}}{\partial x_t^2}$$

so that, exactly as in previous instances, it suffices to show that δ is convex. Letting

$$h(k_t):=\sum_{j=1}^{J}\frac{p_j}{Ak_t+C_j(k_t)}$$
$$g(k_t):=1/h(k_t)$$

we can do the familiar calculation from previous cases to conclude that it suffices to demonstrate

$$2\left(\frac{\partial h}{\partial k_{T-1}}\right)^2<h(k_{T-1})\frac{\partial^2 h}{\partial k_{T-1}^2} \tag{36}$$

To show this we calculate

$$
\begin{aligned}
\frac{\partial h}{\partial k_t} &= -\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2} \\
\frac{\partial^2 h}{\partial k_t^2} &= 2\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})^2}{(Ak_t + C_j(k_t))^3} - \sum_{j=1}^{J} \frac{p_j \frac{\partial^2 C_j}{\partial k_t^2}}{(Ak_t + C_j(k_t))^2}
\end{aligned}
$$

Since $\epsilon^{(t+1)}$ is strictly convex in resources, it follows that the second sum here is positive. Consequently, it's contribution is to make the right side of (36) larger, and hence to complete the proof we must only show that

$$
\left[\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2}\right]^2 \leq \left[\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}\right] \left[\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})^2}{(Ak_t + C_j(k_t))^3}\right]
$$

This inequality is yet again a consequence of the Cauchy-Schwartz inequality, proving convexity.

This closes the induction and completes the proof. □

The main theorem in its full generality adds the additional technical complication that future aggregates appear nonlinearly in the effective discount factor, leading to additional bookkeeping.

Theorem 4: Main Theorem, $\sigma \neq 1$. Make the sequence of recursive definitions

$$
\begin{aligned}
M_T &= (1 - \delta + R_T)^{1-\sigma} \\
Q_{T-1} &= (\beta \mathbb{E}_{T-1} M_T)^{1/\sigma} \\
M_t &= (1 - \delta + R_t)^{1-\sigma}(1 + Q_{t+1})^{\sigma}, \quad && t = 2, ..., T \\
Q_{t-1} &= \left[\beta \mathbb{E}_{t-1} M_t\right]^{1/\sigma}, && t = 2, ..., T
\end{aligned}
$$

The savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ which solve the dynamic programming problems (6) with can be written in the form

$$
\begin{aligned}
k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = {}& \frac{Q_t}{1 + Q_t} x_t - \frac{1}{1 + Q_t} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s}(1 - \delta + R_r)} \right) \\
& + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)
\end{aligned}
$$

where the nonlinear error term $\epsilon^{(t)}$ is strictly decreasing, convex, and satisfies

$$
\lim_{x_t \to \underline{k}_t} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{1}{1 + Q_t} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s}(1 - \delta + R_r)} \right) + \underline{k}_t \right]
$$

and

$$\lim_{x_t \to \infty} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = 0$$

Proof. The proof is again by induction, beginning with the period $T-1$ savings function.

Base Case (T-1): We begin by writing out the expected value of the terminal Euler equation (8). We continue to use the notation established in the log case: for a given state $(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})$ of resources and predictive probabilities at time $T-1$ we let

$$\{(\ell_j, R_j, W_j): \ j = 1, ..., J\}$$

denote the possible time T endowment and price outcomes which are assigned positive predictive probability. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j/(1-\delta+R_j)$. We also introduce the notation

$$D_j = \frac{1}{(1-\delta+R_j)^{1-\sigma}}$$

The terminal Euler equation (8) can be written as

$$\begin{aligned}\frac{1}{(x_{T-1}-k_{T-1})^\sigma} &= \beta \left(\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^\sigma} \right) \\ &= \beta \left(\frac{\sum_{j=1}^{J} p_j \prod_{i \neq j} D_i (k_{T-1}+w_i)^\sigma}{\prod_j D_j (k_{T-1}+w_j)^\sigma} \right)\end{aligned}$$

Taking the reciprocal and then taking σth roots, this gives

$$\left(\frac{\beta}{\prod_j D_j} \right)^{1/\sigma} (x_{T-1} - k_{T-1}) = \frac{\prod_j (k_{T-1}+w_j)}{\left(\sum_{j=1}^{J} p_j \prod_{i \neq j} D_i (k_{T-1}+w_i)^\sigma \right)^{1/\sigma}}$$

Some algebra in the numerator and denominator gives

$$\beta(x_{T-1}-k_{T-1}) = \frac{k_{T-1}^J + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} \left(\sum_j p_j \prod_{i \neq j} D_i (1 + \frac{w_i}{k_{T-1}})^\sigma \right)^{1/\sigma}}$$

To apply Lemma 1 on the right hand side, we compute

$$\lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}^{J}+k_{T-1}^{J-1}\sum_j w_j+o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j\prod_{i\neq j}D_i(1+\frac{w_i}{k_{T-1}})^{\sigma}\right)^{1/\sigma}}-\frac{k_{T-1}}{\left(\sum_j p_j\prod_{i\neq j}D_i\right)^{1/\sigma}}\right)$$
$$=\lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}^{J}\left[1-\frac{1}{\left(\sum_j p_j\prod_{i\neq j}D_i\right)^{1/\sigma}}\left(\sum_j p_j\prod_{i\neq j}D_i(1+\frac{w_i}{k_{T-1}})^{\sigma}\right)^{1/\sigma}\right]+k_{T-1}^{J-1}\sum_j w_j+o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j\prod_{i\neq j}D_i(1+\frac{w_i}{k_{T-1}})^{\sigma}\right)^{1/\sigma}}\right)$$

Considering this as the limit of three separate fractions (one for each term in the numerator), we see that the third converges to 0 and the second converges to

$$\frac{\sum_j w_j}{\left(\sum_j p_j\prod_{i\neq j}D_i\right)^{1/\sigma}}$$

We must therefore calculate the limit of the fraction corresponding to the first term; this simplifies to

$$\lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}\left[1-\frac{1}{\left(\sum_j p_j\prod_{i\neq j}D_i\right)^{1/\sigma}}\left(\sum_j p_j\prod_{i\neq j}D_i(1+\frac{w_i}{k_{T-1}})^{\sigma}\right)^{1/\sigma}\right]}{\left(\sum_j p_j\prod_{i\neq j}D_i(1+\frac{w_i}{k_{T-1}})^{\sigma}\right)^{1/\sigma}}\right)$$

The denominator has limit

$$\left(\sum_j p_j\prod_{i\neq j}D_i\right)^{1/\sigma} \tag{37}$$

so we need only compute the limit of numerator, which we rewrite as

$$\left(\frac{\left[1-\frac{1}{\left(\sum_j p_j\prod_{i\neq j}D_i\right)^{1/\sigma}}\left(\sum_j p_j\left[\prod_{i\neq j}D_i\right]\left(1+\frac{\sum_{i\neq j}w_i}{k_{T-1}}+o(1/k_{T-1})\right)^{\sigma}\right)^{1/\sigma}\right]}{1/k_{T-1}}\right)$$

Both numerator and denominator tend to zero here as k_{T-1} increases without bound, so that we may apply l'Hospital's rule to compute the limit. Taking derivatives, we therefore wish to compute the limit of

$$k_{T-1}^2 \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \left(\sum_j p_j \left[\prod_{i\neq j} D_i\right] (1+o(1))^\sigma\right)^{\frac{1}{\sigma}-1}$$
$$\times \left[\sum_j p_j \left[\prod_{i\neq j} D_i\right] \left(1+\frac{\sum_{i\neq j} w_i}{k_{T-1}} + o(1/k_{T-1})\right)^{\sigma-1} \left(-\frac{\sum_{i\neq j} w_i}{k_{T-1}^2} + o(1/k_{T-1}^2)\right)\right]$$
$$= \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \left(\sum_j p_j \left[\prod_{i\neq j} D_i\right] (1+o(1))^\sigma\right)^{\frac{1}{\sigma}-1}$$
$$\times \left[\sum_j p_j \left[\prod_{i\neq j} D_i\right] (1+o(1))^{\sigma-1} \left(-\sum_{i\neq j} w_i + o(1)\right)\right]$$

After some thought, one sees that the limit of the last expression as $k_{T-1} \to \infty$ is

$$\frac{-\sum_j p_j \left[\prod_{i\neq j} D_i\right] \sum_{i\neq j} w_i}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)}$$

Combining this with (37), we now get

$$\lim_{k_{T-1}\to\infty} \left(\frac{k_{T-1}^J + k_{T-1}^{J-1}\sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}} - \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}\right)$$
$$= \frac{\sum_j w_j}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} - \frac{\sum_j p_j \left[\prod_{i\neq j} D_i\right] \sum_{i\neq j} w_i}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1+1/\sigma}}$$
$$= \frac{\sum_j p_j \left[\prod_{i\neq j} D_i\right] w_j}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1+1/\sigma}}$$
$$= \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j$$

Now Lemma 1 implies that the limit as $k_{T-1} \to \infty$ of

$$-\delta(k_{T-1}) := \frac{\prod_j (k_{T-1}+w_j)}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (k_{T-1}+w_i)^\sigma\right)^{1/\sigma}} - \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}$$
$$- \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j$$

vanishes. It now follows from the terminal Euler equation that we can write

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma}(x_{T-1}-k_{T-1}) = \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} + \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j - \delta(k_{T-1})$$

with $\lim_{k_{T-1}\to\infty} \delta(k_{T-1}) = 0$. Rearranging slightly, we get

$$\left(\beta \sum_j p_j/D_j\right)^{1/\sigma}(x_{T-1}-k_{T-1}) = k_{T-1} + \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j - \delta(k_{T-1})$$

Noting that

$$\left(\beta \sum_j p_j/D_j\right)^{1/\sigma} = Q_{T-1}$$
$$\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l} = \frac{1/D_j}{\sum_k p_k/D_k} = \frac{M_j}{\mathbb{E}_{T-1}M_T}$$

and solving for k_{T-1} gives

$$k_{T-1} = \frac{1}{1+Q_{T-1}}\left(Q_{T-1}x_{T-1} - \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j + \delta(k_{T-1})\right)$$

By the existence and uniqueness proposition, there is a unique period $T-1$ savings function $k^{(T-1)}$ which solves the household problem in this period. Hence we may define

$$\epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) := \delta\left(k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\right)$$

so that we obtain the desired form

$$k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \frac{1}{1+Q_{T-1}}\left(Q_{T-1}x_{T-1} - \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j + \epsilon^{(T-1)}(x_1, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\right)$$

As in other cases, the limit of the savings function at the borrowing constraint is determined from solving

$$\underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \frac{1}{1+Q_{T-1}} \Bigg(Q_{T-1}\underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1}) - \sum_j p_j \left(\frac{\prod_{i \neq j} D_i}{\sum_k p_k \prod_{l \neq k} D_l} \right) w_j$$

$$+ \lim_{x_{T-1} \to \underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1})} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) \Bigg)$$

We also have

$$\lim_{x_{T-1} \to \infty} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \lim_{x_{T-1} \to \infty} \delta \left(k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) \right) = 0$$

since $k^{(T-1)}(x_{T-1}) \to \infty$ as $x_{T-1} \to \infty$ (by the existence and uniqueness proposition) and $\delta \to 0$ as its argument increases to ∞.

To establish that $\epsilon^{(T-1)}$ is decreasing in resources, we once again observe that

$$\frac{\partial \epsilon^{(T-1)}}{\partial x_{T-1}} = \frac{\partial \delta}{\partial k_{T-1}} \frac{\partial k^{(T-1)}}{\partial x_{T-1}}$$

has the same sign as $\partial \delta / \partial k_{T-1}$. Next we observe that the definition of $\delta(k_{T-1})$ satisfies

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}} \right)^{-1/\sigma} = k_{T-1} + \mathbb{E}_{T-1} w_T - \delta(k_{T-1})$$

It therefore suffices to show that the derivative of the left side of the above equality with respect to k_{T-1} is larger than unity. Calculating this derivative, it suffices to show that

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}} \right)^{-\frac{1}{\sigma}-1} \left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma+1}} \right) > 1$$

Rearranging, this is equivalent to the bound

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}} \right)^{1/\sigma} < \left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma+1}} \right)^{\frac{1}{\sigma+1}}$$

That this inequality is true is a consequence of the fact that L^{σ} norms are increasing in σ, once again following from Jensen's inequality.

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(T-1)}}{\partial x_{T-1}^2} = \frac{\partial^2 \epsilon^{(T-1)}}{\partial x_{T-1}^2}$$

so that, exactly as in the two period CRRA case, it suffices to show that δ is convex. Letting

$$h(k_{T-1}) := \sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma}}$$
$$g(k_{T-1}) := 1/h(k_{T-1})$$

and repeating the derivative calculations in that case, we see that it suffices to show that

$$\left(\frac{1}{\sigma}+1\right)\left(\frac{\partial h}{\partial k_{T-1}}\right)^2 - h(k_{T-1})\frac{\partial^2 h}{\partial k_{T-1}^2} < 0$$

To do we calculate

$$\frac{\partial h}{\partial k_{T-1}} = -\sigma \sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+1}}$$
$$\frac{\partial^2 h}{\partial k_{T-1}^2} = \sigma(\sigma+1) \sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+2}}$$

so it suffices to show that

$$\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+1}}\right]^2 - \left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma}}\right]\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+2}}\right] < 0$$

which is equivalent to

$$\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+1}}\right]^2 < \left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma}}\right]\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+2}}\right]$$

This inequality is once again true by the Cauchy-Schwartz inequality, and this establishes our base case for the induction.

Inductive Step. Suppose now that the theorem is proved for the savings function in period $t+1$, where $1 < t \leq T-1$. For a given state $(x_t, \mathcal{L}_t, \mathcal{F}_t)$ of resources and predictive probabilities at time t, we suppose that there are J time $t+1$ states which are assigned positive predictive probability, and we let

$$\{(\ell_j, R_j, W_j) : \ j = 1, ..., J\}$$

denote the possible time $t+1$ endowment and price outcomes. Note that both J and the set in the above line may change depending on which period t we are focused on - we suppress these dependencies for simplicity of notation. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j\ell_j/(1-\delta+R_j)$ and $D_j = \frac{1}{(1-\delta+R_j)^{1-\sigma}} = 1/M_j$.

The inductive step will involve what we already know about the time $t+1$ savings function and its nonlinear error term. For simplicity, we once again suppress the dependence of these functions on predictive distributions in the argument.

Having established these notational conventions the time t Euler equation can be written as

$$\frac{1}{(x_t - k_t)^\sigma} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{D_j \left(k_t + w_j - \frac{1}{1-\delta+R_j} k_{t+1} \right)^\sigma} \right)$$

For optimality, time $t+1$ savings must be given by the time $t+1$ savings function, and by the inductive hypothesis we have

$$\begin{aligned}
&\frac{1}{1-\delta+R_j} k^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big) \\
&\quad = \frac{Q_j}{1+Q_j}(k_t + w_j) - \frac{1}{1+Q_j} \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1-\delta+R_r)} \right) \\
&\quad + \frac{1}{1-\delta+R_j} \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

so that the bracketed expressions in the denominators on the right side of the Euler equation take the form

$$\begin{aligned}
&\frac{1}{1+Q_j} k_t + \frac{1}{1+Q_j} \left(w_j + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1-\delta+R_r)} \right) \right) \\
&\quad - \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

In order to simplify the algebraic steps to follow, we let

$$\begin{aligned}
A_j &:= \frac{1}{1+Q_j} \\
B_j &:= \frac{1}{1+Q_j} \left(w_j + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1-\delta+R_r)} \right) \right) \\
C_j(k_t) &:= B_j - \frac{\epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)}{1-\delta+R_j}
\end{aligned}$$

In this notation, the Euler equation becomes

$$\frac{1}{(x_t - k_t)^\sigma} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{D_j (A_j k_t + C_j(k_t))^\sigma} \right)$$

We now proceed much as in the terminal case, by writing this Euler equation as

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma}(x_t - k_t) = \frac{\prod_j (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}}$$

In order to apply Lemma 1, we now compute

$$\lim_{k_t\to\infty}\left(\frac{\prod_{j=1}^J (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}} - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} k_t\right)$$

We may rewrite the argument here as a single fraction. To do so, we first note that

$$\lim_{k_t\to\infty}\frac{C_j(k_t)}{k_t} = \lim_{k_t\to\infty}\left(\frac{B_j}{k_t} - \frac{\epsilon^{(t+1)}\Big((1-\delta+R_j)k_t + W_j\ell_j\Big)}{k_t}\right) = 0$$

so that $C_j(k_t) = o(k_t)$. We therefore want to compute the limit as $k_t \to \infty$ of

$$\frac{k_t^J\left(\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}\right) + k_t^{J-1}\sum_j C_j(k_t)\prod_{i\neq j} A_j + o(k_t^{J-1})}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}}$$

$$= \frac{k_t^J\left(\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}\right) + k_t^{J-1}\sum_j C_j(k_t)\prod_{i\neq j} A_j + o(k_t^{J-1})}{k_t^{J-1}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}}$$

$$= \frac{k_t\left(\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}\right) + \sum_j C_j(k_t)\prod_{i\neq j} A_j + o(1)}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + o(1))^\sigma\right)^{1/\sigma}}$$

Splitting this into three separate fractions and taking limits termwise, the limit of the third term vanishes and that of the second term is

$$\lim_{k_t\to\infty}\frac{\sum_j C_j(k_t)\prod_{i\neq j} A_j}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + o(1))^\sigma\right)^{1/\sigma}} = \frac{\sum_j B_j \prod_{i\neq j} A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}$$

The limit of the first term requires more attention once again. The denominator approaches

$$\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}$$

while we can use l'Hospital's rule to calculate that of the numerator upon writing it as

$$\frac{\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} \left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}}{1/k_t}$$

and observing that both numerator and denominator tend to 0 as k_t tends to ∞. Taking the necessary derivatives, we must compute the limit of

$$\begin{aligned}
&k_t^2 \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} \frac{1}{\sigma} \left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i \left(A_i + \frac{C_i(k_t)}{k_t}\right)^\sigma\right)^{\frac{1}{\sigma}-1} \\
&\quad \times \Big[\sum_{j=1}^J \sigma p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \left(1 + \sum_{i\neq j} \frac{C_i(k_t)}{A_i k_t} + o(1/k_t)\right)^{\sigma-1} \left(\sum_{i\neq j} \left[\frac{\partial C_i/\partial k_t}{A_i k_t} - \frac{C_i(k_t)}{A_i k_t^2}\right] + o(1/k_t^2)\right)\Big] \\
&= \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} \left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i \left(A_i + o(1)\right)^\sigma\right)^{\frac{1}{\sigma}-1} \\
&\qquad \times \Big[\sum_{j=1}^J p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) (1+o(1))^{\sigma-1} \left(\sum_{i\neq j} \left[\frac{k_t}{A_i}\frac{\partial C_i}{\partial k_t} - \frac{C_i(k_t)}{A_i}\right] + o(1)\right)\Big]
\end{aligned}$$

Taking the limit $k_t \to \infty$ of the last line, we end up with

$$\frac{\prod_j A_j}{\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma} \Big[\sum_{j=1}^J p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \lim_{k_t\to\infty} \left(\sum_{i\neq j} \left[\frac{k_t}{A_i}\frac{\partial C_i}{\partial k_t} - \frac{C_i(k_t)}{A_i}\right]\right)\Big] \tag{38}$$

To calculate the remaining limit, we observe from the following short, technical argument that

$$\lim_{k_t\to\infty} k_t \frac{\partial C_i}{\partial k_t} = 0$$

To see this, first observe that we have

$$\begin{aligned}
\frac{\partial C_i}{\partial k_t} k_t &= \frac{\partial \epsilon^{(t+1)}\big((1-\delta+R_i)k_t + w_i\big)}{\partial x_{t+1}} (1-\delta+R_i)k_t \\
&= \Big((1-\delta+R_i)k_t + w_i\Big) \frac{\partial \epsilon^{(t+1)}\big((1-\delta+R_i)k_t + w_i\big)}{\partial x_{t+1}} \\
&\quad - w_i \frac{\partial \epsilon^{(t+1)}\big((1-\delta+R_i)k_t + w_i\big)}{\partial x_{t+1}}
\end{aligned}$$

Making the change of variables $x_{t+1} = (1-\delta+R_i)k_t + w_i$, this can be written as

$$= x_{t+1} \frac{\partial \epsilon^{(t+1)}}{\partial x_{t+1}} - w_i \frac{\partial \epsilon^{(t+1)}}{\partial x_{t+1}}$$

Then, since $\epsilon^{(t+1)} \to 0$ as $x_{t+1} \to \infty$, the same is true for its derivative, and the second term in the last line above vanishes in this limit. Since $x_{t+1} \to \infty$ and $k_t \to \infty$ it therefore suffices to show that the first term also vanishes in this limit.

Recalling that ϵ^{t+1} is convex in resources, it must be the case that

$$|\epsilon^{(t+1)}(x) - \epsilon^{(t+1)}(x_0)| \geq \left| \frac{\partial \epsilon^{(t+1)}(x_0)}{\partial x_{t+1}} (x - x_0) \right|$$

for any $x_0, x \in (\underline{k}_{t+1}, \infty)$. In particular, we can take $x = 2x_0$ giving

$$|\epsilon^{(t+1)}(2x_0) - \epsilon^{(t+1)}(x_0)| \geq \left| \frac{\partial \epsilon^{(t+1)}(x_0)}{\partial x_{t+1}} (x_0) \right|$$

Taking the limit as $x_0 \to \infty$, the left side vanishes, while the right side remains greater than or equal to zero. It follows from the squeeze theorem that

$$\left| \frac{\partial \epsilon^{(t+1)}(x_0)}{\partial x_{t+1}} x_0 \right| \to 0$$

as needed.

Recalling now the definition of $C_i(k_t)$ and using the above calculation, we arrive at the closed form expression for (38)

$$-\frac{\prod_j A_j}{\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma} \Big[\sum_{j=1}^{J} p_j \left(\prod_{i \neq j} D_i A_i^\sigma \right) \sum_{i \neq j} \frac{B_i}{A_i} \Big]$$

Combining this with the previously calculated limits, we obtain

$$\lim_{k_t \to \infty} \left(\frac{\prod_{j=1}^{J} (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^{J} p_j \prod_{i \neq j} D_i (A_i k_t + C_i(k_t))^\sigma \right)^{1/\sigma}} - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1/\sigma}} k_t \right)$$
$$= \frac{\sum_j B_j \prod_{i \neq j} A_j}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1/\sigma}} - \frac{\prod_j A_j \Big[\sum_{j=1}^{J} p_j \left(\prod_{i \neq j} D_i A_i^\sigma \right) \frac{B_i}{A_i} \Big]}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right) \left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1/\sigma}}$$

Combining the fractions and simplifying, we get

$$\frac{\left(\sum_j B_j \prod_{i \neq j} A_j \right) \left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right) - \prod_j A_j \Big[\sum_{j=1}^{J} p_j \left(\prod_{i \neq j} D_i A_i^\sigma \right) \sum_{i \neq j} \frac{B_i}{A_i} \Big]}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1 + \frac{1}{\sigma}}}$$

Some thought allows us to rearrange this as

$$\left(\prod_j A_j\right) \frac{\sum_j p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \frac{B_j}{A_j}}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}}$$

Lemma 1 now tells us that

$$\begin{aligned}-\delta(k_t) := \frac{\prod_{j=1}^J (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}} - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} k_t \\ - \left(\prod_j A_j\right) \frac{\sum_j p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \frac{B_j}{A_j}}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}}\end{aligned}$$

vanishes in the limit $k_t \to \infty$. It follows that we can rewrite the time t Euler equation as

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma} (x_t - k_t) = \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} k_t + \frac{\left(\prod_j A_j\right) \sum_j p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \frac{B_j}{A_j}}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}} - \delta(k_t)$$

with $\lim_{k_t\to\infty} \delta(k_t) = 0$. Rearranging somewhat, we get

$$\begin{aligned}x_t - k_t &= \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} k_t + \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i A_i^\sigma}{\sum_l p_l \prod_{m\neq l} D_m A_m^\sigma}\right) \frac{B_j}{A_j} - \delta(k_t) \\ &= \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} k_t + \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} \sum_j p_j \left(\frac{1/(D_j A_j^\sigma)}{\sum_l p_l/(D_l A_l^\sigma)}\right) \frac{B_j}{A_j} - \delta(k_t)\end{aligned}$$

or

$$k_t = \frac{1}{1 + \left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} \left(\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma} x_t - \sum_j p_j \left(\frac{1/(D_j A_j^\sigma)}{\sum_l p_l/(D_l A_l^\sigma)}\right) \frac{B_j}{A_j} + \delta(k_t)\right)$$

Recalling our choice of notation D_j and A_j, we see that this can now be written as

$$k_t = \frac{1}{1+Q_t} \left(Q_t x_t - \mathbb{E}_t \left(\frac{M_{t+1}}{\mathbb{E}_t M_{t+1}}\right) \frac{B_{t+1}}{A_{t+1}} + \delta(k_t)\right)$$

We note that the definitions of B_j and A_j give

$$\frac{B_j}{A_j} = \left(w_j + \mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r}\right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1 - \delta + R_r)}\right)\right)$$

so this further reduces to

$$k_t = \frac{1}{1+Q_t}\left(Q_t x_t - \mathbb{E}_t \sum_{s=t+1}^{T}\left(\prod_{r=t+1}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)} + \delta(k_t)\right)$$

As in previous arguments, we define

$$\epsilon^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t) := \delta\left(k^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t)\right)$$

which gives us the desired form of the savings function.

The limits of the nonlinear error follow as usual: the limit at the borrowing constraint from rearranging the identity

$$\begin{aligned}\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t) = \frac{1}{1+Q_t}\Bigg(&Q_t\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t) - \mathbb{E}_t \sum_{s=t+1}^{T}\left(\prod_{r=t+1}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\\ &+ \lim_{x_t\to\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t)}\epsilon^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t)\Bigg)\end{aligned}$$

and the limit at ∞ from the definition of $\epsilon^{(t)}$ above, the behavior of $k^{(t)}$ at ∞, and the definition of δ.

By a similar argument to previous cases, monotonicity of the error term will follow from showing that the derivative of

$$\left(\sum_{j=1}^{J}\frac{p_j}{D_j(A_jk_t + C_j(k_t))^{\sigma}}\right)^{-1/\sigma}$$

with respect to k_t is larger than

$$\frac{1}{\left(\sum_j p_j/(D_jA_j^{\sigma})\right)^{1/\sigma}}$$

Taking the derivative, this amounts to showing that

$$\left(\sum_{j=1}^{J}\frac{p_j}{D_j(A_jk_t + C_j(k_t))^{\sigma}}\right)^{-\frac{1}{\sigma}-1}\left(\sum_{j=1}^{J}\frac{p_j(A_j+\frac{\partial C_j}{\partial k_t})}{D_j(A_jk_t + C_j(k_t))^{\sigma+1}}\right) > \frac{1}{\left(\sum_j p_j/(D_jA_j^{\sigma})\right)^{1/\sigma}}$$

Noting that

$$\frac{\partial C_j}{\partial k_t} = -(1-\delta+R_j)\frac{\partial\epsilon^{(t+1)}}{\partial x_{t+1}} > 0$$

it is sufficient to show that

$$\left(\sum_{j=1}^{J}\frac{p_j}{D_j(A_jk_t+C_j(k_t))^\sigma}\right)^{\frac{1}{\sigma}+1}\leq\left(\sum_{j=1}^{J}\frac{p_jA_j}{D_j(A_jk_t+C_j(k_t))^{\sigma+1}}\right)\left(\sum_j p_j/(D_jA_j^\sigma)\right)^{1/\sigma}$$

which is implied by showing

$$\sum_{j=1}^{J}\frac{p_j}{D_jA_j^\sigma(k_t+C_j(k_t)/A_j)^\sigma}\leq\left(\sum_{j=1}^{J}\frac{p_j}{D_jA_j^\sigma(k_t+C_j(k_t)/A_j)^{\sigma+1}}\right)^{\frac{\sigma}{\sigma+1}}\left(\sum_j p_j/(D_jA_j^\sigma)\right)^{\frac{1}{1+\sigma}}$$

This last inequality is true by Hölder's inequality.

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(t)}}{\partial x_t^2}=\frac{\partial^2\epsilon^{(t)}}{\partial x_t^2}$$

so that, exactly as in previous instances, it suffices to show that δ is convex. Letting

$$h(k_t):=\sum_{j=1}^{J}\frac{p_j}{D_j(A_jk_t+C_j(k_t))^\sigma}$$
$$g(k_t):=h(k_t)^{-1/\sigma}$$

we can do the familiar calculation from previous cases to conclude that it suffices to demonstrate that

$$\left(\frac{1}{\sigma}+1\right)\left(\frac{\partial h}{\partial k_{T-1}}\right)^2<h(k_{T-1})\frac{\partial^2 h}{\partial k_{T-1}^2}\tag{39}$$

To show this we calculate

$$\frac{\partial h}{\partial k_t}=-\sigma\sum_{j=1}^{J}\frac{p_j(A_j+\frac{\partial C_j}{\partial k_t})}{D_j(A_jk_t+C_j(k_t))^{\sigma+1}}$$
$$\frac{\partial^2 h}{\partial k_t^2}=\sigma(\sigma+1)\sum_{j=1}^{J}\frac{p_j(A_j+\frac{\partial C_j}{\partial k_t})^2}{D_j(A_jk_t+C_j(k_t))^{\sigma+2}}-\sigma\sum_{j=1}^{J}\frac{p_j\frac{\partial^2 C_j}{\partial k_t^2}}{D_j(A_jk_t+C_j(k_t))^{\sigma+1}}$$

Since $\epsilon^{(t+1)}$ is strictly convex in resources, it follows that the second sum here is strictly positive. Consequently, it's contribution is to make the right side of (39) larger, and hence to complete the proof we must only show that

$$\left[\sum_{j=1}^{J}\frac{p_j(A_j+\frac{\partial C_j}{\partial k_t})}{D_j(A_jk_t+C_j(k_t))^{\sigma+1}}\right]^2\leq\left[\sum_{j=1}^{J}\frac{p_j}{D_j(A_jk_t+C_j(k_t))^\sigma}\right]\left[\sum_{j=1}^{J}\frac{p_j(A_j+\frac{\partial C_j}{\partial k_t})^2}{D_j(A_jk_t+C_j(k_t))^{\sigma+2}}\right]$$

This inequality is yet again a consequence of the Cauchy-Schwartz inequality, proving convexity.

This closes the induction and completes the proof. □

7 Appendix B: Description of Algorithm

Time is discrete and infinite. In each period, a measure $1/T$ of T-period lived households is born, so that at any given time there is a unit measure of living individuals. Households behave according to the theory described in Section 3. Equilibrium for this economy will now involve a distribution $\Lambda_s(k, \ell, t)$ at time s of households across capital k, employment ℓ, and age $t = 1, ..., T$. For each time s, we initialize the new cohort by fixing $\Lambda_s(0, 0, 1) = u_s/T$ and $\Lambda_s(0, 1, 1) = (1 - u_s)/T$, where u_s is the average unemployment rate corresponding to the aggregate state at time s. Households born into unemployment are given a small unemployment benefit (μ), which is paid by taxing the wage of current workers. This overlapping-generations, life-cycle framework has several advantages. First, as $T \to \infty$, it replicates well-known environments (Aiyagari (1994), Krusell and Smith (1998)). Second, for $T = 2$, we are able to study an exact equilibrium; that is, one that is not subject to the curse of dimensionality. This facilitates a precise evaluation of common numerical approaches. Third, it is a natural environment for breaking aggregation as households representing different generations have different marginal propensities to save, as documented above. Finally, an infinite horizon economy permits the study of simulation methods and steady state analysis. Thus, we can use the KS algorithm described above to solve the model. Suppose the current aggregate state is good $z = g$; young, employed households solve:[21]

$$\begin{aligned}
&\max_{k_t} \ln(W_t L_t \ell_t^t (1-\tau_t) - k_t) \\
&+\beta\pi_{gg}\left(\frac{\pi_{gg11}}{\pi_{gg}}\ln(W_{t+1}^{(g)}L_{t+1}^{(g)}\ell_{t+1}^t(1-\tau_{t+1}^{(g)}) + R_{t+1}^{(g)}k_t) + \frac{\pi_{gg10}}{\pi_{gg}}\ln(\mu + R_{t+1}^{(g)}k_t)\right) \\
&+\beta\pi_{gb}\left(\frac{\pi_{gb11}}{\pi_{gb}}\ln(W_{t+1}^{(b)}L_{t+1}^{(b)}\ell_{t+1}^t(1-\tau_{t+1}^{(b)}) + R_{t+1}^{(b)}k_t) + \frac{\pi_{gb10}}{\pi_{gb}}\ln(\mu + R_{t+1}^{(b)}k_t)\right)
\end{aligned} \tag{40}$$

Young, unemployed households solve:

$$\begin{aligned}
&\max_{k_t} \ln(\mu - k_t) \\
&+\beta\pi_{gg}\left(\frac{\pi_{gg01}}{\pi_{gg}}\ln(W_{t+1}^{(g)}L_{t+1}^{(g)}\ell_{t+1}^t(1-\tau_{t+1}^{(g)}) + R_{t+1}^{(g)}k_t) + \frac{\pi_{gg00}}{\pi_{gg}}\ln(\mu + R_{t+1}^{(g)}k_t)\right) \\
&+\beta\pi_{gb}\left(\frac{\pi_{gb01}}{\pi_{gb}}\ln(W_{t+1}^{(b)}L_{t+1}^{(b)}\ell_{t+1}^t(1-\tau_{t+1}^{(b)}) + R_{t+1}^{(b)}k_t) + \frac{\pi_{gb00}}{\pi_{gb}}\ln(\mu + R_{t+1}^{(b)}k_t)\right)
\end{aligned} \tag{41}$$

The current old consume according to, $c_t^{t-1} = R_t^{(g)}k_{t-1} + W_t^{(g)}L_t^{(g)}\ell_t^{t-1}(1-\tau_t)$ if employed and $c_t^{t-1} = R_t^{(g)}k_{t-1} + \mu$ if unemployed.

The auctioneer must only provide $W_{t+1}^{(s)}, R_{t+1}^{(s)}$ for each state $s = \{g, b\}$ along with the corre-

[21] We adopt the notation that superscripts denote date of birth. For example, those born at t, get employment share ℓ_{t+1}^t at date $t+1$. Aggregate states are denoted as superscripts in parenthesis.

sponding aggregate state probabilities to the current young. The following algorithm describes an "exact auctioneer."

Algorithm 1: Exact Auctioneer

1) Draw a series of aggregate shocks Z_t, $t = 1...T$ for the simulation. Fix an initial distribution of old households at $t = 1$ by placing a mass $u(Z_1)/2$ at a value $k_{0,1}$ and a mass $(1 - u(Z_1))/2$ at a value $k_{1,1}$. Define initial aggregate capital

$$K_1 = \frac{1}{2}(u(Z_1)k_{0,1} + (1 - u(Z_1))k_{1,1}); \tag{42}$$

2) Do the following for each $t = 1, ..., T - 1$:

   a) Guess a value K_{guess} for K_{t+1}.
   b) Given K_t and the guessed value for K_{t+1}, calculate R_t, W_t, R_{t+1}, W_{t+1}.
   c) Given the prices from the previous step, solve the household's problem for the savings $k_{0,t+1}$ and $k_{1,t+1}$ of unemployed and employed (respectively) young households at time t.
   d) Let

$$K_{\text{imp}} = \frac{1}{2}(u(Z_t)k_{0,t+1} + (1 - u(Z_t))k_{1,t+1}) \tag{43}$$

   If K_{imp} is equal to K_{guess} within tolerance, let $K_{t+1} = K_{\text{guess}}$, increase t and return to $2.a$. Otherwise, update the guess and return to $2.b$.

7.1 The KS Auctioneer

1. Guess an initial savings function, aggregate law of motion, and cross-sectional distribution of households. Generate a long sequence of total factor productivity shocks once and for all.

2. Solve the household's problem by Euler equation iteration, beginning from the initial savings function and using the aggregate law of motion to forecast one-period-forward prices. Iterate until the savings functions converge up to some tolerance.

3. Use the savings function from Step 3 to simulate the cross-sectional distribution for the sequence of TFP shocks generated in Step 1 via the procedure of Young (2010).

4. Use the time series of distributional statistics generated in Step 3 to update the aggregate law of motion, for example by ordinary least squares regression (in the case of a law which is linear in coefficients).

5. Repeat steps 2-4 until the aggregate law of motion converges within some tolerance.

6. Test for equilibrium. For example, one can compute the R squared fit of the regression in Step 4, or use the procedure of den Haan (2010).

We follow Maliar, Maliar, and Valli (2010) in our grid choices: with deterministic steady state capital given by

$$K_{ss} = \left(\frac{\frac{1}{\beta} - (1 - \delta)}{\alpha} \right)^{-\frac{1}{1-\alpha}}$$

the idiosyncratic grid points are distributed on the interval $[0, 25K_{ss}]$ according to the polynomial rule

$$k_j = \left(\frac{j}{100} \right)^7 25 \cdot K_{ss}, \quad j = 1, ..., 100$$

while four aggregate grids are distributed linearly on the interval $[0.8K_{ss}, 1.3K_{ss}]$. Parameters were chosen in line with the computational literature: $\beta = 0.99$, $\alpha = 0.36$, $\delta = 0.025$, and $\sigma = 1$. Productivity shocks take values in the set $\{0.99,\ 1.01\}$ and efficiency shocks take values in the set $\{0,\ 1.111\}$, and these shocks follow the joint Markov process of Krusell and Smith (1998). We do not include taxes or unemployment insurance for this exercise.[22]

[22] The algorithm was implemented in the programming language Julia and calculations performed on an MSI GT70 2QD Laptop with an Intel Core i7-4710MQ processor and 16 GB of RAM. Interpolation in step 2 uses cubic splines via the Julia package Dierckx, which simply acts as a wrapper for the FORTRAN package of the same name. The model solved in 43.22 seconds, with 34 iterations on the aggregate law of motion and 1909 iterations on the Euler equation for the initial aggregate law of motion loop.